Amsterdam University College

Logic, Information flow and Argumentation

Homework exercises, Week 2, part b (due Tuesday 21 February).

1. For each of the following formulas, decide whether it is a tautology, a contradiction, or neither (in which case it is obviously satisfiable).

   (a) $p \leftrightarrow q$,

   (b) $p \wedge \neg p$,

   (c) $((p \to q) \to p) \to p$

   (d) $p \to (p \wedge q)$

   (e) $(p \to q) \to (\neg q \to \neg p)$,

   (f) $(p \wedge (p \to q)) \wedge \neg p$,

   (g) $\neg\neg p \to p$.

2. Decide whether the following inference is valid or not. If not, specify a counter-example.

   (a) $(p \vee q) \vee r \models p \vee (q \vee r)$,

   (b) $p \to q \models q \to p$,

   (c) $\neg p \models p \to q$,

   (d) $\{p \to q, q \to r\} \models p \to r$,

   (e) $\{p \to q, p \to r\} \models p \to (q \wedge r)$.

3. In the following exercise, you are asked to show a number of *general facts* about propositional logic and the notion of logical consequence. The formulas φ, ψ, φ_i etc. stand for *arbitrary* formulas in the language of propositional logic.

   (a) Suppose $\{\varphi_1, \ldots, \varphi_n\}$ is any collection of premises, and ψ is a tautology. Show that $\{\varphi_1, \ldots, \varphi_n\} \models \psi$ is valid.

   (b) Suppose φ is a tautology, and suppose $\varphi \models \psi$ is valid. Show that then ψ is also a tautology.

(c) Suppose φ is a contradiction, and ψ is an arbitrary formula. Show that $\varphi \models \psi$ is valid (hint: think about counterexamples).

(d) Suppose φ is not a contradiction, but ψ is. Show that $\varphi \models \psi$ is invalid.

(e) Let Φ denote the empty collection of formulas (an empty set). Note that for each valuation V, the statement "V satisfies all formulas in Φ" is trivially true, since there *aren't* any formulas in Φ.

Now show that a formula ψ is a tautology if and only if $\Phi \models \psi$, i.e., if and only if it is a logical consequence of the empty set of premises (this explains the notation "$\models \psi$" for tautologies.)

(f) Suppose that $\varphi \models \psi$ is a valid inference and $\psi \models \chi$ is a valid inference. Show that then $\varphi \models \chi$ is also a valid inference (*transitivity*).

(g) Suppose that $\varphi \models \psi$ is a valid inference. Show that $\varphi, \chi \models \psi$ is also a valid inference (*monotonicity*).

(h) Let φ and ψ be arbitrary formulas. Show that the inference $\varphi \models \psi$ is valid if and only if $\varphi \to \psi$ is a tautology. (If you want an extra challenge, you can show a stronger version: $\{\varphi_1, \ldots, \varphi_n\} \models \psi$ if and only if $\{\varphi_1, \ldots, \varphi_{n-1}\} \models \varphi_n \to \psi$. This is called the *deduction theorem*.)

N.B. When we want to prove that 'A if and only if B', we must prove both directions of the bi-implication. That is, our proof has two parts: 1) a proof of 'if A then B', and 2) a proof of 'if B then A'.