Chapter 9

Proofs

In the first part of this book we have discussed complete axiomatic systems for propositional and predicate logic. In the previous chapter we have introduced the tableau systems of Beth, which was a method to test validity. This method is much more convenient to work with since it tells you exactly what to do when a given formula has to be dealt with during such a validity test. Despite the convenience of Beth's system it does not represent the way humans argue.

In the late 1930s the German mathematician Gerhard Gentzen developed a system which he called *natural deduction*, in which the deduction steps as made in mathematical proofs are formalized. This system is based not so much on axioms but on rules instead. For each logical symbol, connectives and quantifiers, rules are given just in the way they are dealt with in mathematical proofs.

Gerhard Gentzen

Dag Prawitz

In this chapter we will demonstrate how this system works. The precise definition of these rules goes back to the Swedish logician Dag Prawitz, who gave a very elegant reformulation of Gentzen's work in the 1960s.[1]

9.1 Natural deduction for propositional logic

In chapter 2 we have introduced an axiomatic system for propositional logic. By means of the rule of modus ponens one may jump from theorems to new theorems. In addition we had three axioms, theorems that do not have to be proven, and which may be used as starting points. Since these axioms are tautologies, and the rule of modus ponens is a sound rule of inference, a proof is then just a list of tautologies, propositions that are true under all circumstances.

Although this system is fun to work with for enthusiast readers who like combinatoric puzzles, it is surely not the way people argue. You may remember that it took us even five steps to prove that an extremely simple tautology as $\varphi \rightarrow \varphi$ is valid. This may even be worse for other trivial cases. It takes almost a full page to prove that $\varphi \rightarrow \neg\neg\varphi$ is valid (a real challenge for the fanatic puzzler)!

The pragmatic problem of a purely axiomatic system is that it does not facilitate a transparent manner of *conditional reasoning*, which makes it deviate very much from human argumentation. In an ordinary setting people derive conclusions which hold under *certain* circumstances, rather than summing up information which always hold. Especially when conditional propositions, such as the implicative formulas as mentioned here above, have to be proven the conditionals are used as presuppositions. Let us illustrate this with a simple mathematical example.

> If a square of a positive integer doubles the square of another positive integer then these two integers must both be even.
>
> Suppose m, n are two positive integers such that $m^2 = 2n^2$. This means m must be even, because if m^2 is even then m must be even as well. So, $m = 2k$ for some positive integer k. Since $m^2 = 2n^2$ we get $2n^2 = (2k)^2 = 4k^2$, and therefore, $n^2 = 2k^2$ which means that n must be even as well.

In the proof we presuppose that the antecedent ($m^2 = 2n^2$) of the conditional proposition ($m^2 = 2n^2 \rightarrow m, n$ even) that is to be proven holds. This is what is called an *hypothesis*. In the proof we derived that the consequent (m, n even) of the proposition holds under the circumstances that the hypothesis holds. The validity of this type of conditional reasoning reflects an important formal property of propositional logic (and also of the other logics

[1]The format of the proofs in this chapter has been introduced by the American logician John Fitch. Prawitz used tree like structures, whereas here, in analogy of Fitch's presentation proofs are divided into so-called subproofs.

which has been introduced in the first part of this book), which is called the *deduction property*. Formally it looks as follows: For every set of formulas Σ and for every pair of formulas φ and ψ:

$$\Sigma, \varphi \models \psi \text{ if and only if } \Sigma \models \varphi \to \psi \tag{9.1}$$

It says that by means of the implication we can reason about valid inference within the propositional language explicitly. A conditional proposition $\varphi \to \psi$ is a valid inference within a context Σ if and only if ψ is a valid conclusion from Σ extended with φ as an additional assumption (hypothesis). The deduction property reveals the operational nature of implication: φ leads to the conclusion ψ.

Exercise 9.1 Show that this deduction property holds for propositional logic by making use of truth tables.

Exercise 9.2 The modus ponens rule and the deduction property are characteristic for the implication in propositional logic. Let ⓒ be some propositional connective which has the modus ponens and deduction property:

$$\varphi, \varphi \copyright \psi \models \psi \qquad \varphi \models \psi \text{ if and only if } \models \varphi \copyright \psi$$

Show that ⓒ must be the implication $\to$.

Integration of the deduction property in a deduction system requires accommodation of hypotheses, i.e., additional assumptions that a reasoner uses in certain parts of his line of argumentation or proof. A proof of $\varphi \to \varphi$ then becomes trivial. Since assuming φ leads to φ we may conclude that $\varphi \to \varphi$ is always true. We may write this as follows:

$$\begin{array}{ll} \left[\begin{array}{ll} \underline{\quad\varphi\quad} & \\ \varphi & {\scriptstyle\text{repeat}} \end{array}\right] & \\ \varphi \to \varphi & {\scriptstyle\text{Ded}} \end{array} \tag{9.2}$$

The first part between square brackets we call a *subproof* of the full proof. A subproof starts with an hypothesis (underlined) which is assumed to hold within this subproof. Proving a conditional proposition $\varphi \to \psi$ requires a subproof with hypothesis φ and conclusion ψ within this subproof. For our simple example (9.2) this immediately leads to success, but it may involve much more work for longer formulas. Consider the following example where we want to prove the second axiom of the axiomatic system as given in chapter 2: $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.

$$\begin{array}{ll} \left[\begin{array}{l} \underline{\quad\varphi \to (\psi \to \chi)\quad} \\ \vdots \\ (\varphi \to \psi) \to (\varphi \to \chi) \end{array}\right] & \\ (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) & {\scriptstyle\text{Ded}} \end{array} \tag{9.3}$$

We have first set up a preliminary format of our proof. The conditional proposition that we want to prove has been rewritten as a subproof, which we have to establish later on. We need to show that the antecedent of the proposition indeed leads to the consequent. Since the desired conclusion of the subproof is an implication again we may follow the same procedure and extend our first format in the following way:

$$\begin{array}{c} \left[\begin{array}{c} \varphi \to (\psi \to \chi) \\ \hline \left[\begin{array}{c} \varphi \to \psi \\ \hline \vdots \\ \varphi \to \chi \end{array}\right] \\ (\varphi \to \psi) \to (\varphi \to \chi) \quad {\scriptstyle \text{Ded}} \end{array}\right] \\ (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \quad {\scriptstyle \text{Ded}} \end{array} \tag{9.4}$$

Here we have a subproof within a subproof, in which we need to show that the additional assumption $\varphi \to \psi$ leads to a conclusion $\varphi \to \chi$. This second hypothesis has been added to the hypothesis of the first subproof. In order to obtain the desired conclusion we may therefore use both hypotheses.

Again, the conclusion is a conditional proposition, and so, for the third time, we squeeze in a new subproof.

$$\begin{array}{c} \left[\begin{array}{c} \varphi \to (\psi \to \chi) \\ \hline \left[\begin{array}{c} \varphi \to \psi \\ \hline \left[\begin{array}{c} \varphi \\ \hline \vdots \\ \chi \end{array}\right] \\ \varphi \to \chi \quad {\scriptstyle \text{Ded}} \end{array}\right] \\ (\varphi \to \psi) \to (\varphi \to \chi) \quad {\scriptstyle \text{Ded}} \end{array}\right] \\ (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \quad {\scriptstyle \text{Ded}} \end{array} \tag{9.5}$$

Given this reformulation, we need to prove that χ holds given three hypotheses: $\varphi \to (\psi \to \chi)$, $\varphi \to \psi$ and φ. This is not very hard to prove by making use of our earlier rule of modus ponens. From the second and the third ψ follows and from the first and the third $\psi \to \chi$. These new propositional formulas can then be combined to establish χ. Here is our final result:

$$
\begin{array}{l}
\left[\begin{array}{l}
\varphi \to (\psi \to \chi) \\
\hline
\left[\begin{array}{l}
\varphi \to \psi \\
\hline
\left[\begin{array}{ll}
\varphi & \\
\hline
\psi & {\scriptstyle \text{MP}} \\
\psi \to \chi & {\scriptstyle \text{MP}} \\
\chi & {\scriptstyle \text{MP}}
\end{array}\right] \\
\varphi \to \chi \quad {\scriptstyle \text{Ded}}
\end{array}\right] \\
(\varphi \to \psi) \to (\varphi \to \chi) \quad {\scriptstyle \text{Ded}}
\end{array}\right] \\
(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \quad {\scriptstyle \text{Ded}}
\end{array}
\tag{9.6}
$$

This result means that we no longer have to use the second axiom of the axiomatic system as described in chapter 2. It can be derived by means of our new deduction rule.

The first axiom of the system, $\varphi \to (\psi \to \varphi)$, can be established also quite straightforwardly by means of the deduction rule. In order to prove $\varphi \to (\psi \to \varphi)$ we need to show that $\psi \to \varphi$ can be proved from φ. This can be shown then by simply concluding that φ follows from φ and ψ:

$$
\begin{array}{l}
\left[\begin{array}{l}
\varphi \\
\hline
\left[\begin{array}{ll}
\psi & \\
\hline
\varphi & {\scriptstyle \text{rep.}}
\end{array}\right] \\
\psi \to \varphi \quad {\scriptstyle \text{Ded}}
\end{array}\right] \\
\varphi \to (\psi \to \varphi) \quad {\scriptstyle \text{Ded}}
\end{array}
\tag{9.7}
$$

Exercise 9.3 Prove $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$.

9.1.1 Proof by refutation

It seems that we can replace the axiomatic system by a natural deduction system by simply replacing the axioms by a single rule, the deduction rule. This is not the case, however. The third axiom of the axiomatic system $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$, also called *contraposition*, can not be derived by deduction and modus ponens only. We need something to deal with the negations in this formula.

There seems to be a way out by taking $\neg\varphi$ to be an abbreviation of the conditional formula $\varphi \to \bot$. This establishes a procedure to prove negative information by means of the deduction rule. Proving $\neg\varphi$ requires a proof that the assumption that φ holds leads to a contradiction ($\bot$). This is indeed a natural way to establish negative information, as shown in the following example

> $\sqrt{2}$ is not a rational number.
>
> Suppose $\sqrt{2}$ were a rational number. This means there are two positive integers m and n such that $(m/n)^2 = 2$ and, in addition, that m or n is odd, since we can simply take the smallest pair such that $(m/n)^2 = 2$ (they cannot both be even since then it would not be the smallest pair for which this equation holds). But then $m^2 = 2n^2$ and therefore m and n must be even, as we have shown in an earlier example (page 9-2). Clearly, we have derived a contradiction, and therefore $\sqrt{2}$ must be an irrational number.

A reformulation in natural deduction style looks as follows:

$$\begin{array}{l} \left[\begin{array}{l} \sqrt{2}\in\mathbb{Q} \\ \hline (m/n)^2 = 2 \text{ for certain pair of positive integers} \\ m,n \text{ with } m \text{ or } n \text{ being odd.} \\ m = 2n^2 \\ m \text{ and } n \text{ are both even positive integers} \\ \bot \end{array}\right] \\ \neg(\sqrt{2}\in\mathbb{Q}) \end{array} \tag{9.8}$$

This way of proving negative statements suffices to derive certain propositional logical theorems containing negative information. For example, the converse of the contraposition axiom can be established in this way;

$$\begin{array}{l} \left[\begin{array}{l} \varphi\to\psi \\ \hline \left[\begin{array}{l} \psi\to\bot \\ \hline \left[\begin{array}{l} \varphi \\ \hline \psi \quad {\scriptstyle \text{MP}} \\ \bot \quad {\scriptstyle \text{MP}} \end{array}\right] \\ \varphi\to\bot \quad {\scriptstyle \text{Ded}} \end{array}\right] \\ (\psi\to\bot)\to(\varphi\to\bot) \quad {\scriptstyle \text{Ded}} \end{array}\right] \\ (\varphi\to\psi)\to((\psi\to\bot)\to(\varphi\to\bot)) \quad {\scriptstyle \text{Ded}} \end{array} \tag{9.9}$$

Replacing $\to\bot$ by negations then settles $(\varphi\to\psi)\to(\neg\psi\to\neg\varphi)$. Unfortunately, this simple solution does not work for the axiom of contraposition. To get a complete system we need an additional rule.

Exercise 9.4 Show, by trying out the procedure which we have used for the previous examples, that you can not derive the axiom of contraposition by modus ponens and the deduction rule only.

This supplementary rule that we will need is in fact quite close to the deduction rule for negations. To derive a formula $\neg\varphi$ we prove that φ leads to a contradiction, which in fact says that φ can not be true. Our new rule says that φ can be proven by showing that φ can not be *false*. In terms of subproofs, if the hypothesis $\neg\varphi$ leads to a contradiction we may conclude that φ is the case. In this way we can prove the contraposition indeed.

$$
\begin{array}{l}
\left[\begin{array}{ll}
1.\ \neg\varphi \to \neg\psi & \\ \hline
\left[\begin{array}{ll}
2.\ \psi & \\
\left[\begin{array}{ll}
3.\ \neg\varphi & \\ \hline
4.\ \neg\psi & \text{MP 1,3} \\
5.\ \bot & \text{MP 2,5}
\end{array}\right] & \\
6.\ \varphi & \text{new rule 3-5}
\end{array}\right] & \\
7.\ \psi \to \varphi & \text{Ded 2-6}
\end{array}\right] \\
8.\ (\neg\varphi \to \neg\psi) \to (\psi \to \varphi) \quad \text{Ded 1-7}
\end{array}
\tag{9.10}
$$

In step 6 we derived φ from the subproof $[\neg\varphi \mid \ldots \bot]$. The hypothesis that φ is false has led to a contradiction. The contradiction in 5 is obtained by a modus ponens, since $\neg\psi$ is an abbreviation of $\psi \to \bot$ here.

Exercise 9.5 Prove $((\varphi \to \psi) \to \varphi) \to \varphi$. Despite the absence of negations in this formula, you will need the new rule.

Exercise 9.6 Prove $\varphi \to \neg\neg\varphi$ and $\neg\neg\varphi \to \varphi$. Which of those proofs makes use of the new rule?

In logic this new rule is also called *proof by refutation*, or more academically, *reductio ad absurdum*. In fact, it captures the same way of reasoning as we have used in the tableau systems of the previous chapter. Proving the validity of an inference by presenting a closed tableau we show that the given formula can never be false, and therefore must be true, under the circumstances that the premises hold.

9.1.2 Introduction and elimination rules

The three rules suffice to obtain a complete system for propositional logic. The tradition in natural deduction is to separate the treatment of negations and implications which leads to the following five rules.

$$
\begin{array}{ccccc}
 & \vdots & & \vdots & \vdots \\
\vdots & \left[\begin{array}{c}\varphi \\ \hline \vdots \\ \psi \\ \vdots\end{array}\right] & \vdots & \left[\begin{array}{c}\varphi \\ \hline \vdots \\ \bot \\ \vdots\end{array}\right] & \left[\begin{array}{c}\neg\varphi \\ \hline \vdots \\ \bot \\ \vdots\end{array}\right] \\
\varphi \to \psi & & \neg\varphi & & \\
\vdots & & \vdots & & \\
\varphi & & \varphi & & \\
\vdots & & \vdots & & \\
\psi \quad {\scriptstyle \mathrm{E}\to} & \vdots & \bot \quad {\scriptstyle \mathrm{E}\neg} & \vdots & \vdots \\
 & \varphi \to \psi \quad {\scriptstyle \mathrm{I}\to} & & \neg\varphi \quad {\scriptstyle \mathrm{I}\neg} & \varphi \quad {\scriptstyle \mathrm{E}\bot}
\end{array}
\tag{9.11}
$$

These rules are called *elimination* (E) and *introduction* (I) rules. The modus ponens is called an elimination rule since it says how to remove an implication $\varphi \to \psi$ and replace it by its consequent ψ. The rule then obtains the structural name E$\to$. Elimination of negation, E$\neg$, is then, as a consequence, the derivation of $\bot$ from $\neg\varphi$ and φ. The introduction rule for implication, I$\to$, is the deduction rule because it puts an implication on stage. I$\neg$ is defined analogously. The last rule represents the rule of proof by refutation and is most often seen as elimination of $\bot$ (E$\bot$).[2]

Two simpler versions of the deduction rule and the rule of proof by refutation are sometimes added to the system such that repetitions, as for example in the proof of $\varphi \to (\psi \to \varphi)$ as given in (9.7), can be avoided. If a statement φ is true then it also holds under arbitrary conditions: $\psi \to \varphi$. This is in fact a variation of the deduction rule (without hypothesis).

$$
\begin{array}{l}
\left[\begin{array}{ll}\varphi & \\ \hline \psi \to \varphi & {\scriptstyle \mathrm{I}\to\text{ ‘simple’}}\end{array}\right] \\
\varphi \to (\psi \to \varphi) \qquad {\scriptstyle \mathrm{I}\to}
\end{array}
\tag{9.12}
$$

For proofs of refutation the analogous simplification is called *ex falso*. Everything may be derived from a contradiction. We will use these simplified versions also in the sequel of this chapter. In general deductive form they look as follows:

$$
\begin{array}{cc}
\vdots & \vdots \\
\psi & \bot \\
\vdots & \vdots \\
\varphi \to \psi \quad {\scriptstyle \mathrm{I}\to} & \varphi \quad {\scriptstyle \mathrm{E}\bot}
\end{array}
\tag{9.13}
$$

[2]Sometimes I$\neg$ is used for this rule, and then the introduction rule for negation is called falsum introduction (I$\bot$).

9.1.3 Rules for conjunction and disjunction

In propositional logic we also want to have rules for the other connectives. We could try the same procedure as we have done for negation. Find an equivalent formulation in terms of $\rightarrow$ and $\bot$ and then derive rules for these connectives.

$$\varphi \vee \psi \equiv (\varphi \rightarrow \bot) \rightarrow \psi \qquad \varphi \wedge \psi \equiv (\varphi \rightarrow (\psi \rightarrow \bot)) \rightarrow \bot \tag{9.14}$$

This option does not lead to what may be called a system of natural deduction. The equivalent conditional formulas are much too complicated. Instead, we use direct rules for manipulating conjunctive and disjunctive propositions. Below the introduction and elimination rules are given for the two connectives.

$$\boxed{\begin{array}{cccc}
\begin{array}{ll}
\vdots & \\
\varphi \wedge \psi & \\
\vdots & \\
\varphi/\psi & \text{I}\vee
\end{array}
&
\begin{array}{ll}
\vdots & \\
\varphi & \\
\vdots & \\
\psi & \\
\vdots & \\
\varphi \wedge \psi & \text{I}\wedge
\end{array}
&
\begin{array}{ll}
\vdots & \\
\varphi \vee \psi & \\
\vdots & \\
\left[\begin{array}{c} \varphi \\ \hline \vdots \\ \chi \\ \vdots \end{array}\right] & \\
\vdots & \\
\left[\begin{array}{c} \psi \\ \hline \vdots \\ \chi \\ \vdots \end{array}\right] & \\
\vdots & \\
\chi & \text{E}\vee
\end{array}
&
\begin{array}{ll}
\vdots & \\
\varphi/\psi & \\
\vdots & \\
\varphi \vee \psi & \text{I}\vee
\end{array}
\end{array}} \tag{9.15}$$

The rules for conjunction are quite straightforward. The elimination of a conjunction is carried out by selecting one of its arguments. Since we know that they are both true this is perfectly sound and a natural way of eliminating conjunctions. Introduction of a conjunction is just as easy. Derive a conjunction if both arguments have already been derived.

The introduction of a disjunction is also very simple. If you have derived one of the arguments then you may also derive the disjunction. The rule is perfectly correct but it is not very valuable, since in general, the disjunction contains less information then the information conveyed by one of the arguments.

The elimination of the disjunction is the most complicated rule. It uses two subproofs, one for each of the arguments of the disjunction. If in both subproofs, starting with one of the disjuncts (φ, ψ) as a hypothesis, the same information can be derived (χ) then we know that this must also hold in a context in which we are uncertain which of the arguments in fact holds ($\varphi \vee \psi$). Despite the complexity of the rule, its soundness can be seen quite easily. We leave this to the reader in the next exercise.

Exercise 9.7 Show that $\Sigma, \varphi \vee \psi \models \chi$ if and only if $\Sigma, \varphi \models \chi$ and $\Sigma, \psi \models \chi$.

The elimination rule of disjunction reflects a natural way of dealing with uncertainty in argumentation. Here is an example of a mathematical proof.

> There exists two irrational numbers x and y such that x^y is rational.
>
> Let $z = \sqrt{2}^{\sqrt{2}}$. This number must be either irrational *or* rational. Although, we are uncertain about the status of z we can find in both cases two irrational x and y such that x^y must be rational.
>
> Suppose that z is rational, then we may take $x = y = \sqrt{2}$. We have just seen earlier that $\sqrt{2}$ is irrational, so this choice would be satisfactory.
>
> Suppose that z is irrational. Then we may take $x = z$ and $y = \sqrt{2}$, because then $x^y = z^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$, and that is a perfect rational number.

In the deduction style we could reformulate our argumentation as follows

$$
\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \vee \sqrt{2}^{\sqrt{2}} \notin \mathbb{Q} \\
\left[\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \\ \hline
x = y = \sqrt{2} \\
x^y = \sqrt{2}^{\sqrt{2}} \\
x^y \in \mathbb{Q} \text{ for certain } x, y \notin \mathbb{Q}
\end{array}\right] \\
\left[\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q} \\ \hline
x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2} \\
x^y = 2 \\
x^y \in \mathbb{Q} \text{ for certain } x, y \notin \mathbb{Q}
\end{array}\right] \\
x^y \in \mathbb{Q} \text{ for certain } x, y \notin \mathbb{Q}
\end{array}
\tag{9.16}
$$

In practical reasoning disjunction elimination is also manifest as a way to jump to conclusions when only uncertain information is available. The following realistic scenario gives an illustration of this.

> I am traveling from A to B by train. If I run to the railway station of my home town A then I'll be in time to catch the train to B at 7.45AM, and then in B I will take the bus to the office and I will be there in time. If I won't run then I won't catch the 7.45AM train, but in this case I could take the train to B at 8.00AM instead. I would then need to take a cab from the railway station in B to arrive in time at the office. I start running to the railway station, not being sure whether my physical condition this morning will be enough to make me catch the first train (last night I have been to the cinema, and later on we went to the pub, etcetera). But no worries, I'll be in time at the office anyway (okay, it will cost me a bit more money if I won't catch the first train, since taking a cab is more expensive then taking the bus).

Here is the deductive representation of my reasoning:

$$\begin{array}{l} \text{Catch 7.45AM-train} \;\vee\; \text{Catch 8.00AM-train} \\ \left[\begin{array}{l} \text{Catch 7.45AM-train.} \\ \hline \text{Take the bus in B to the office.} \\ \text{I'll be in time at the office.} \end{array}\right] \\ \left[\begin{array}{l} \text{Catch 8.00AM-train.} \\ \hline \text{Take a cab in B to the office.} \\ \text{I'll be in time at the office.} \end{array}\right] \\ \text{I'll be in time at the office.} \end{array} \tag{9.17}$$

Exercise 9.8 Can you make up a similar scenario, jumping to safe conclusion while being uncertain about the conditions, from your personal daily experience? Now, reformulate this as a deduction such as given for the previous example.

Here is a very simple example of disjunction elimination in propositional logic. We derive $\psi \vee \varphi$ from the assumption $\varphi \vee \psi$:

$$\begin{array}{l} 1.\quad \varphi \vee \psi \\ \left[\begin{array}{l} 2.\quad \varphi \\ \hline 3.\quad \psi \vee \varphi \quad {\scriptstyle \mathrm{I}\vee\, 2} \end{array}\right] \\ \left[\begin{array}{l} 4.\quad \psi \\ \hline 5.\quad \psi \vee \varphi \quad {\scriptstyle \mathrm{I}\vee\, 4} \end{array}\right] \\ 6.\quad \psi \vee \varphi \quad {\scriptstyle \mathrm{E}\vee\, 1,2\text{-}3,4\text{-}5} \end{array} \tag{9.18}$$

The formula $\psi \vee \varphi$ can be derived from φ and ψ by applying I$\vee$, so we can safely conclude $\psi \vee \varphi$ from $\varphi \vee \psi$ by E$\vee$.

Exercise 9.9 Prove $\varphi \to \psi$ from the assumption $\neg\varphi \vee \psi$.

Exercise 9.10 Prove $\neg(\varphi \wedge \psi)$ from $\neg\varphi \vee \neg\psi$.

Exercise 9.11 Prove $(\varphi \vee \psi) \wedge (\varphi \vee \chi)$ from $\varphi \vee (\psi \wedge \chi)$.

In general, disjunction elimination applies whenever we need to prove a certain formula χ from a disjunctive assumption $\varphi \vee \psi$. The strength of the elimination rule for disjunction is reflected by the equivalence of the inference $\varphi \vee \psi \models \chi$ on the one hand and the two inferences $\varphi \models \chi$ and $\psi \models \chi$ on the other (as you may have computed for yourself when you have made exercise 9.7 on page 9-10).

Disjunctive conclusions are much harder to establish in a deduction because of the earlier mentioned weakness of the introduction rule for disjunctions. Direct justification of a conclusion $\varphi \vee \psi$ by means of I$\vee$ requires a proof of one of the arguments, φ or ψ, which in many cases is simply impossible. Often a refutational proof is needed to obtain the desired disjunctive conclusion, that is, we show that $\neg(\varphi \vee \psi)$ in addition to the assumptions leads to a contradiction ($\bot$).

A clear illustration can be given by one of the most simple tautologies: $\varphi \vee \neg\varphi$. When it comes to reasoning with truth-values the principle simply says that there are only two opposite truth-values, and therefore it is also called 'principle of the excluded third' or 'tertium non datur'. From an operational or deductive point of view the truth of $\varphi \vee \neg\varphi$ is much harder to see. Since, in general, φ and $\neg\varphi$ are not tautologies, we have to prove that $\neg(\varphi \vee \neg\varphi)$ leads to a contradiction. Below a deduction, following the refutational strategy, has been given:

$$\begin{array}{lll}
 & \left[\begin{array}{lll}
1 & \neg(\varphi \vee \neg\varphi) & \\ \hline
 & \left[\begin{array}{lll} 2 & \varphi & \\ \hline 3 & \varphi \vee \neg\varphi & \text{I}\vee\ 2 \\ 4 & \bot & \text{E}\neg\ 1{,}3 \end{array}\right] & \\
5 & \neg\varphi & \text{I}\neg\ 2\text{-}4 \\
6 & \varphi \vee \neg\varphi & \text{I}\vee\ 5 \\
7 & \bot & \text{E}\!\to 1{,}6
\end{array}\right] & \\
8 & \varphi \vee \neg\varphi & \bot_{\text{E}}\ 1\text{-}7
\end{array} \qquad (9.19)$$

As you can see $\neg\varphi$ is derived from $\neg(\varphi \vee \neg\varphi)$ and this gives us finally the contradiction that we aimed at. In general this is the way to derive a disjunctive conclusion $\varphi \vee \psi$ for which a direct proof does not work. We assume the contrary $\neg(\varphi \vee \psi)$ then derive $\neg\varphi$ or $\neg\psi$ (or both) and show that this leads to a contradiction.

Exercise 9.12 Prove by a deduction that $(\varphi \to \psi) \vee (\psi \to \varphi)$ is a tautology.

Exercise 9.13 Deduce $\neg\varphi \vee \neg\psi$ from $\neg(\varphi \wedge \psi)$

Exercise 9.14 Prove by a deduction that $\neg\varphi \vee \psi$ follows from $\varphi \to \psi$.

9.2 Natural deduction for predicate logic

The natural deduction system for predicate logic consists of two simple rules and two more complicated, but at the same time more compelling, rules for the quantifiers $\forall$ and $\exists$. The easy weaker rules are $\forall$-elimination and $\exists$-introduction. They are just generalizations of the earlier elimination rule for $\wedge$ and the introduction rule for $\vee$.

From $\forall x\,\varphi$ we may derive that φ holds for 'everything'. This means that we substitute a term for x in φ. Substitution only has a small syntactic limitation. A term may contain variables, and we have to take care that no variable which occurs in such a 'substitute' gets bound by a quantifier in φ after replacing the occurrence of x by this term. If this is the case we say that this term is *substitutable* for x in φ. As an illustration that things go wrong when we neglect this limitation take the formula $\forall x \exists y\, \neg(x = y)$. Obviously, this formula is true in every model with more than one object. If we substitutes y for x in $\exists y\, \neg(x = y)$ we get $\exists y\, \neg(y = y)$ which is an inconsistent formula. The term y is not substitutable since y gets bound by the existential quantifier in $\exists y\, \neg(x = y)$.

Introduction of the existential quantifier works in the same way. If you have derived a property φ for certain term t you may replace this term by x and derive $\exists x\,\varphi$ successively. If we write $\varphi[t/x]$ for the result of substitution of t for x in φ and in addition prescribing that t must be substitutable for x in φ, we can formulate the rules mentioned here as follows:

$$\boxed{\begin{array}{lll} \vdots & & \vdots \\ \forall x\,\varphi & & \varphi[t/x] \\ \vdots & & \vdots \\ \varphi[t/x] \quad \scriptstyle\mathrm{E}\forall & & \exists x\,\varphi \quad \scriptstyle\mathrm{I}\exists \end{array}} \tag{9.20}$$

In practice these weak rules are only used to make small completing steps in a proof.

Also in the condition of I$\exists$ it is required that t is substitutable for x in φ. To see that neglecting this additional constraint leads to incorrect result take the formula $\forall y\, y = y$. This is a universally valid formula. It is also the result of replacing x by y in the formula $\forall y\, x = y$, but $\exists x \forall y\, x = y$ is certainly not a valid consequence of $\forall y\, y = y$: $\exists x \forall y\, x = y$ only holds in models containing only a single object.

The introduction rule of the universal quantifier is a bit more complicated rule, but, at the same time, it is a very strong rule. The rule is also referred at as *generalization*. By proving that a property φ holds for an *arbitrary* object we derive that φ holds for *all* objects: $\forall x\,\varphi$. To make sure that the object of which we prove φ-ness is indeed completely arbitrary we use a new name which is not a constant in the language. Starting the subproof we extend the language with this new name only within the range of this subproof. Such an additional constant is also called a *parameter*. It may not be used in the main line of the proof which contains this subproof. Here is the formal version of the rule I∀.

$$\begin{array}{|l|}\hline \vdots \\ \left[\begin{array}{l} \hfill c \\ \hline \vdots \\ \varphi[c/x] \end{array}\right] \\ \vdots \\ \forall x\,\varphi \qquad\qquad \text{I}\forall \\ \hline \end{array} \tag{9.21}$$

As you can see the subproof does not contain an hypothesis. The only information which is relevant here is that the parameter c does not appear in the line of the argument outside the subproof (represented by the vertical dots outside the subproof box), and that it is not a constant in the base language. To stress this minor syntactical limitation we indicate this c on top of the line where the subproof starts. This makes it clear that this is the reference to the arbitrary object for which we have to prove the desired property φ.

In natural settings of argumentation the generalization rule is most often combined with the deduction rule (I→). If the assumption that an arbitrary object has the property φ leads to the conclusion that it also must have another property ψ we have proven that 'All φ are ψ' or in predicate logical notation $\forall x\,(\varphi \to \psi)$. In a formal deduction this looks as follows:

$$\begin{array}{l} \left[\begin{array}{l} \hfill c \\ \hline \left[\begin{array}{l} \varphi[c/x] \\ \hline \vdots \\ \psi[c/x] \end{array}\right] \\ (\varphi \to \psi)[c/x] \qquad \text{I}{\to} \end{array}\right] \\ \forall x\,(\varphi \to \psi) \qquad \text{I}\forall \end{array} \tag{9.22}$$

If we are able to prove for an arbitrary man that he must be mortal, we have proven that all men are mortal.

Take, as a mathematical example of this combination, the proof that if the square of a positive integer m doubles the square of another positive integer n, $m^2 = 2n^2$, they must both be even (page 9-2). The generalization rule, applied twice, would then rephrase this as universal result (given that the domain of discourse here contains only positive integers)

$$\forall x \forall y\,(x^2 = 2y^2 \to x, y \text{ both even})$$

Here is a first example deduction in predicate logic, showing that $\forall x\,(Px \wedge Qx)$ follows from two assumptions, $\forall x\, Px$ and $\forall x\, Qx$:

$$
\begin{array}{lll}
1. & \forall x\, Px & \text{Ass} \\
2. & \forall x\, Qx & \text{Ass} \\
 & \left[\begin{array}{lll} 3. & & c \\ \hline 4. & Pc \quad \text{E}\forall\ 1 & \\ 5. & Qc \quad \text{E}\forall\ 2 & \\ 6. & Pc \wedge Qc \quad \text{I}\wedge\ 4,5 & \end{array}\right] & \\
7. & \forall x\,(Px \wedge Qx) \quad \text{I}\forall\ 3\text{-}6 &
\end{array}
\tag{9.23}
$$

As you can see the generalization rule dominates the proof. It determines the external structure of the proof, whereas the weaker rule E$\forall$ shows up only within the very inner part of the proof. The generalization rule works pretty much the same way as the deduction rule in propositional logic. In pure predicate logic a proof of a universal statement requires most often the generalization procedure. As we will see in the next section, there are other rules to prove statements with universal strength when we apply predicate logic for reasoning about a specific mathematical structure: the natural numbers.

Just as I$\exists$ is a generalization of I$\vee$, the elimination of existential quantifiers is taken care of by a generalization of disjunction elimination. A formula $\exists x\, \varphi$ represents that there is an object which has the property φ but, in general, we do not know who or what this φ-er is. To jump to a fixed conclusion we introduce an arbitrary φ-er, without caring about who or what this φ-er is, and show that this is enough to derive the conclusion that we are aiming at. The rule looks as follows:

$$
\boxed{
\begin{array}{ll}
\vdots & \\
\exists x\, \varphi & \\
\vdots & \\
\left[\begin{array}{ll} \varphi[c/x] & c \\ \hline \vdots & \\ \psi & \end{array}\right] & \\
\vdots & \\
\psi & \text{E}\exists
\end{array}
}
\tag{9.24}
$$

The conclusion ψ can be derived on the basis of the introduction of an arbitrary φ-er (and nothing else). This means that if such a φ-er exists ($\exists x\, \varphi$) then we may safely conclude that ψ must hold. Again, the indication of the parameter c reminds us that it restricted by the same limitations as in the generalization rule I$\forall$.

There is also a close relation with the generalization and the deduction rule. Combination of the latter two facilitated a way to prove statements of the form ‘All φ are ψ’. In fact a slight variation of this is presupposed by means of the subproof in the E$\exists$- rule. Here it in fact says that it has been proven for an arbitrary object that if this object has the property φ then ψ must hold. And then we conclude that, given the assumption that there exists such a φ-er ($\exists x\, \varphi$), we know that ψ must hold.

The following variant of the train scenario as discussed on page 9-11 illustrates elimination of uncertainty conveyed by existential information in practice.

> Again, I am traveling from A to B. I don’t know when trains leave, but I know at least there is a train departing from A going to B every half hour. Right now it is 7.35AM, and it will take me only ten minutes to get to the station. This means that I’ll catch some train before 8.15AM: or *some point in time t* between 7.45AM and 8.15AM. The train from A to B takes 35 minutes, and my arrival at B will therefore be before 8.50AM (t + 35’ $<$ 8.50AM). A cab ride will bring me in less than 15 minutes to the office and so I will be at the office before 9.05AM (t + 35’ + 15’ $<$ 9.05AM). This means I will be there before 9.15AM, when the first meeting of this morning starts.

Although I am quite uncertain about the time of departure I can safely conclude that I will be in time at the office.

Below a deduction is given which proves that $\forall x \exists y\, Rxy$ follows from $\exists y \forall x\, Rxy$. Each of the quantifier rules is used once:

$$
\begin{array}{l}
1.\ \ \exists y \forall x\, Rxy \qquad \text{Ass} \\
\left[\begin{array}{ll}
2.\ \ \forall x\, Rxc & c \\ \hline
\left[\begin{array}{ll}
3. & d \\ \hline
4.\ \ Rdc & \text{E}\forall\ 2 \\
5.\ \ \exists y\, Rdy & \text{I}\exists\ 4
\end{array}\right] & \\
6.\ \ \forall x \exists y\, Rxy & \text{I}\forall\ 3\text{-}5
\end{array}\right] \\
7.\ \ \forall x \exists y\, Rxy \qquad \text{E}\exists\ 2\text{-}6
\end{array}
\tag{9.25}
$$

As an explanation what the individual steps in this proof mean, let us say that Rxy stands for ‘x knows y’ in some social setting. The assumption says there is some ‘famous’ person known by everybody. The conclusion that we want to derive means that ‘everybody knows someone’. We started with E$\exists$, introducing an arbitrary person known by everybody, and we called him or her c ($\forall x\, Rxc$), and from this we want to derive the same conclusion ($\forall x \exists y\, Rxy$). To get this settled, we introduced an arbitrary object d and proved that d must know somebody ($\exists y\, Rdy$). This is proved by using Rdc (I$\exists$) which follows from $\forall x\, Rxc$ (E$\forall$).

Let us try a more difficult case: $\forall x\, Px \lor \exists x\, Qx$ follows from $\forall x\,(Px \lor Qx)$. The conclusion is a disjunction, and one can easily see that both disjuncts are not valid consequences of the given assumption. This means we have to prove this by using refutation (E$\bot$). We need to show that the assumption and the negation of the conclusion lead to a contradiction ($\bot$). Here is the complete deduction.

$$
\begin{array}{lll}
1. & \forall x\,(Px \lor Qx) & \text{Ass}\\
\hline
 & \quad 2.\ \neg(\forall x\, Px \lor \exists x\, Qx) & \\
 & \quad\quad 3.\ \qquad\qquad c & \\
 & \quad\quad 4.\ Pc \lor Qc & \text{E}\forall\ 1\\
 & \quad\quad\quad 5.\ Pc & \\
 & \quad\quad\quad 6.\ Qc & \\
 & \quad\quad\quad 7.\ \exists x\, Qx & \text{I}\exists\ 6\\
 & \quad\quad\quad 8.\ \forall x\, Px \lor \exists x\, Qx & \text{I}\exists\ 7\\
 & \quad\quad\quad 9.\ \bot & \text{E}\neg\ 2,8\\
 & \quad\quad\quad 10.\ Pc & \text{E}\bot\ 9 \text{ (simple version)}\\
 & \quad\quad 11.\ Pc & \text{E}\lor\ 4,5\text{-}5,6\text{-}10\\
 & \quad 12.\ \forall x\, Px & \text{I}\forall\ 3\text{-}11\\
 & \quad 13.\ \forall x\, Px \lor \exists x\, Qx & \text{I}\lor\ 12\\
 & \quad 14.\ \bot & \text{E}\neg\ 2,13\\
15. & \forall x\, Px \lor \exists x\, Qx & \text{E}\bot\ 2\text{-}14
\end{array}
\tag{9.26}
$$

In the outermost subproof we have shown that $\forall x\,(Px \lor Qx)$ in combination with $\neg(\forall x\, Px \lor \exists x\, Qx)$ leads to the conclusion $\forall x\, Px$ (12) which gives indeed an immediate contradiction (13,14). $\forall x\, Px$ can be obtained by proving the property P for an arbitrary object (c), which is carried in the second subproof. This can be proved then by using $Pc \lor Qc$ and disjunction elimination. Pc immediately follows from the first disjunct, Pc itself, and it also follows from Qc, since this leads to a contradiction and by applying ex falso, the simple form of proof by refutation, also to Pc.

Exercise 9.15 Prove that $\forall x\, \neg Px$ follows from $\neg\exists x\, Px$.

Exercise 9.16 Prove that $\exists x\, Px \land \exists x\, Qx$ follows from $\exists x\,(Px \land Qx)$.

Exercise 9.17 Prove that $\exists x\, \neg Px$ follows from $\neg\forall x\, Px$. You need to prove this by refutation, since a direct proof of the existential conclusion is not possible.

Exercise 9.18 Prove that $\exists x\,(Px \lor Qx)$ follows from $\exists x\, Px \lor \exists x\, Qx$, and also the other way around.

Exercise 9.19 (*) Prove that $\exists x\,(Px \to \forall x\, Px)$ is valid. This one requires a proof by refutation as well: show that $\bot$ follows from $\neg\exists x\,(Px \to \forall x\, Px)$.

9.2.1 Rules for identity

In addition to the rules for the quantifiers we also have to formulate rules for identity which are particularly important for mathematical proofs. The introduction rule is the simplest of all rules. It just states that an object is always equal to itself. It is in fact an axiom, there are no conditions which restrict application of this rule.

$$\begin{array}{ll} \vdots & \\ t = t & {\scriptstyle \mathrm{I=}} \end{array} \tag{9.27}$$

The elimination rule says that we always may replace terms by other terms which refer to the same object. We only have to take care that the variables which occur within these terms do not mess up the binding of variables by quantifiers. The term that we replace may only contain variables that occur freely (within the formula which is subject to the replacement), and the substitute may not contain variables which get bound after replacement. If these condition hold then we may apply the following rule:

$$\begin{array}{ll} \vdots & \\ t_1 = t_2 / t_2 = t_1 & \\ \vdots & \\ \varphi & \\ \vdots & \\ \varphi' & {\scriptstyle \mathrm{E=}} \end{array} \tag{9.28}$$

where φ' is the result of replacing occurrences of t_1 by t_2 in φ (not necessarily all). Here are two simple examples showing the symmetry and transitivity of equality:

$$\begin{array}{llllll} 1. & a = b & {\scriptstyle \mathrm{Ass}} & 1. & a = b & {\scriptstyle \mathrm{Ass}} \\ 2. & a = a & {\scriptstyle \mathrm{I=}} & 2. & b = c & {\scriptstyle \mathrm{Ass}} \\ 3. & b = a & {\scriptstyle \mathrm{E=\,1,2}} & 3. & a = c & {\scriptstyle \mathrm{E=\,1,2}} \end{array} \tag{9.29}$$

In the first derivation the first occurrence of a in 2 is replaced by b. In the second derivation the occurrence of b in 2 is replaced by a.

9.3 Natural deduction for natural numbers

In chapter 4 an axiomatic system of arithmetic, as introduced by the Italian logician and mathematician Giuseppe Peano, in predicate logical notation has been discussed.

Giuseppe Peano

In this section we want to give a natural deduction format for Peano's arithmetic, as an example of 'real' mathematical proof by means of natural deduction. These kind of systems are used for precise formalization of mathematical proofs, such that they can be checked, or sometimes be found (that is much harder of course), by computers.

Let us first repeat the axioms as discussed in chapter 4.

$$\begin{array}{ll} P1. & \forall x\,(x+0=x) \\ P2. & \forall x\,\forall y\,(x+sy=s(x+y)) \\ P3. & \forall x\,(x\cdot 0=0) \\ P4. & \forall x\,\forall y\,(x\cdot sy=x\cdot y+x) \\ P5. & \neg\exists x\; sx=0 \\ P6. & (\varphi[0/x]\wedge\forall x\,(\varphi\rightarrow\varphi[sx/x]))\rightarrow\forall x\,\varphi \end{array} \tag{9.30}$$

A straightforward manner to build a predicate logical system for arithmetic is to add these axioms to the system as has been introduced in the previous section. For the first five axioms we do not have an alternative. These axioms are then treated as rules without conditions, and can therefore be applied at any time at any place in a mathematical proof.

The last axiom, the principle of induction, can be reformulated as a conditional rule of deduction, in line with the way it is used in mathematical proofs. For the reader who is not familiar with the induction principle, the following simple example clarifies how it works.

> For every natural number n the sum of the first n odd numbers equals n^2.
>
> For 0 this property holds in a trivial way. The sum of the first zero odd numbers is an empty sum and therefore equals 0, which is also 0^2.
>
> Suppose that the property holds for a certain number k (induction hypothesis).
>
> $$1+3+...+(2k-1)=k^2$$

We need to prove that under this condition the property must also hold for $k+1$ (sk). The sum of the first $k+1$ odd numbers is the same as

$$1 + 3 + ... + (2k-1) + (2k+1)$$

According the induction hypothesis, this must be equal to

$$k^2 + 2k + 1$$

and this equals $(k+1)^2$.

We have proven the property for 0 and also shown that if it holds for a certain natural number then it must also hold for its successors. From this we derive by induction that the property holds for all natural numbers.

9.3.1 The rule of induction

The inductive axiom in Peano's system can be rephrased as a conditional rule in the following way.

$$\begin{array}{ll}
\vdots & \\
\varphi[0/x] & \\
\vdots & \\
\left[\begin{array}{ll}
\varphi[c/x] & \text{c} \\
\hline
\vdots & \\
\varphi[sc/x] & \\
\vdots &
\end{array}\right] & \\
\vdots & \\
\forall x\, \varphi & \text{Ind}
\end{array} \tag{9.31}$$

It mimics the format as has been described by the proof example here above. The formula $\varphi[0/x]$ says that the property φ holds for 0. The subproof represents the inductive step, and starts with the induction hypothesis. We assume $\varphi[c/x]$, i.e., an arbitrary φ-er represented by the parameter c (the induction hypothesis). If this assumption suffices to derive that the successor of c, sc, also must have the property φ, $\varphi[sc/x]$, then φ must hold for all objects, i.e., all the natural numbers.

In terms of the natural deduction system for predicate logic, the induction rule is an additional introduction rule for the universal quantifier. For some cases we can do without this rule and use the generalization rule instead. Here is a simple example which proves that $x + 1$ coincides with the successor sx of x.

$$
\begin{array}{l}
\left[\begin{array}{ll}
1. & \hfill c \\
\hline
2. & \forall x \forall y\, (x + sy = s(x + y)) \quad \text{P2} \\
3. & c + s0 = s(c + 0) \quad \text{E}\forall\ 2\ \text{(twice)} \\
4. & \forall x\, (x + 0 = x) \quad \text{P1} \\
5. & c + 0 = c \quad \text{E}\forall\ 4 \\
6. & c + s0 = sc \quad \text{E}{=}\ 5{,}3
\end{array}\right] \\
7. \quad \forall x\, (x + s0 = sx) \quad \text{I}\forall\ 1\text{-}6
\end{array}
\tag{9.32}
$$

The proof demonstrates that this arithmetical theorem is a pure predicate logical consequence of the two first axioms of the Peano system.

In other cases we have to rely on the induction rule to derive a universal statement about the natural numbers. Here is a very simple example:

$$
\begin{array}{l}
1. \quad 0 + 0 = 0 \quad \text{E}\forall\ \text{P1} \\
\quad \left[\begin{array}{ll}
2. & 0 + c = c \hfill c \\
\hline
3. & 0 + sc = s(0 + c) \quad \text{E}\forall\ \text{P2 (twice)} \\
4. & 0 + sc = sc \quad \text{E}{=}\ 2{,}3
\end{array}\right] \\
5. \quad \forall x\, (0 + x = x) \quad \text{Ind}\ 1{,}2\text{-}4
\end{array}
\tag{9.33}
$$

$0 + x = x$ is the property we have proved for all natural numbers x. First we have shown this is true for 0 and then in the induction step, the subproof 2-4, we have shown that the property $0 + c = c$ for an arbitrary c leads to $0 + sc = sc$.

Exercise 9.20 Prove that $\forall x\, (x \cdot s0 = x)$.

Exercise 9.21 Prove that $\forall x\, (0 \cdot x = 0)$.

The following proof uses both rules, generalization *and* induction.

$$
\left[\begin{array}{lll}
1. & & c \\
\hline
2. & c + s0 = sc & \text{E}\forall\ (9.32) \\
3. & sc + 0 = sc & \text{E}\forall\ \text{P1} \\
4. & c + s0 = sc + 0 & \text{E}{=}\ 3,2 \\
 & \left[\begin{array}{lll}
5. & c + sd = sc + d & d \\
\hline
6. & c + ssd = s(c + sd) & \text{E}\forall\ \text{P2} \\
7. & c + ssd = s(sc + d) & \text{E}{=}\ 5,6 \\
8. & sc + sd = s(sc + d) & \text{E}\forall\ \text{P2} \\
9. & c + ssd = sc + sd & \text{E}{=}\ 8,7
\end{array}\right] & \\
10. & \forall y\,(c + sy = sc + y) & \text{Ind } 4,5\text{-}9
\end{array}\right]
$$
$$
11. \quad \forall x \forall y\,(x + sy = sx + y) \qquad \text{I}\forall\ 1\text{-}10 \tag{9.34}
$$

The outermost subproof justifies the use of a generalization in the last step, whereas the inner subproof contains the induction step of the inductive proof of 10.

Exercise 9.22 Prove that $\forall x\, \forall y\,(x + y = y + x)$ by filling in the gap represented by the vertical dots. You may use the theorem which has already been proved in (9.34).

$$
\left[\begin{array}{lll}
1. & & c \\
\hline
 & \vdots & \\
n. & c + 0 = 0 + c & \ldots \\
 & \left[\begin{array}{lll}
n+1. & c + d = d + c & d \\
\hline
 & \vdots & \\
n+k. & c + sd = sd + c & \ldots
\end{array}\right] & \\
n+k+1. & \forall y\,(c + y = y + c) & \text{Ind } n, n+1\text{-}n+k
\end{array}\right]
$$
$$
n+k+2. \quad \forall x \forall y\,(x + y = y + x) \qquad \text{I}\forall\ 1\text{-}n+k+1
$$

Exercise 9.23 Prove that $\forall x\,(x \cdot ss0 = x + x)$.

Exercise 9.24 (*) Prove that $\forall x \forall y\,(x \cdot y = y \cdot x)$.

Exercise 9.25 Prove that $\forall x \forall y \forall z\,(x + (y + z) = (x + y) + z)$.

9.4 Outlook

9.4.1 Completeness and incompleteness

9.4.2 Natural deduction, tableaus and sequents

9.4.3 Intuitionistic logic

9.4.4 Automated deduction