### Logic in Action

#### Chapter 4: The World according to Predicate Logic

http://www.logicinaction.org/

Statement	Propositional translation
John reads	
John walks	

Statement	Propositional translation
John reads John walks	p

Statement	Propositional translation	
John reads	$oldsymbol{p}$	
John walks	$oldsymbol{q}$	

Statement	Propositional translation
John reads	$oldsymbol{p}$
John walks	q

But the fact that both statements talk about "John" is lost.

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John reads	$oldsymbol{p}$
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With a language including predicates ...

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John walks	q

But the fact that both statements talk about "John" is lost.

With a language including predicates . . .

Statement	Predicate translation
John reads John walks	

Statement	Propositional translation
John reads	$\boldsymbol{p}$
John walks	q

But the fact that both statements talk about "John" is lost.

With a language including predicates . . .

Statement	Predicate translation
John reads John walks	Rj

Statement	Propositional translation
John reads	$oldsymbol{p}$
John walks	$oldsymbol{q}$

But the fact that both statements talk about "John" is lost.

With a language including predicates . . .

Statement	Predicate translation
John reads	Rj
John walks	Wj

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The language of **predicate logic** allow us

- to talk about objects, their properties and their relations with other objects, and
- 2 to make use of **universal** and **existential** quantification.

• Symbols for **constants**:

 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$ 

Symbols for constants:

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$$

2 Symbols for variables:

$$oldsymbol{x},oldsymbol{y},oldsymbol{z},\dots$$

• Symbols for constants:

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$$

2 Symbols for variables:

$$oldsymbol{x},oldsymbol{y},oldsymbol{z},\dots$$

3 Symbols for **predicates**:

$$A, B, C, \dots P, Q, R, \dots$$

Symbols for constants:

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$$

2 Symbols for variables:

$$oldsymbol{x},oldsymbol{y},oldsymbol{z},\dots$$

Symbols for predicates:

$$A, B, C, \ldots P, Q, R, \ldots$$

4 Logical operators:

$$\neg, \land, \lor, \rightarrow, \leftrightarrow$$

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Symbols for predicates:

$$A, B, C, \ldots P, Q, R, \ldots$$

4 Logical operators:

$$\neg, \land, \lor, \rightarrow, \leftrightarrow$$

Quantifiers

$$\forall x \text{ ("for all } x\text{")} \quad \text{and} \quad \exists x \text{ ("there exists an } x\text{")}$$



All A are B

$$\forall x (Ax \to Bx)$$

$$\forall x (Ax \rightarrow Bx)$$

All A are B

 $\forall x (Ax \to Bx)$ 

Some A are B

 $\exists x (Ax \wedge Bx)$ 

$$\forall x (Ax \to Bx)$$

$$\exists x (Ax \land Bx)$$

All A are B

Some A are B

All A are not B (No A is B)

$$\forall x (Ax \rightarrow Bx)$$

$$\forall x (Ax \rightarrow Bx)$$

$$\exists x (Ax \wedge Bx)$$

$$\forall x(Ax o \neg Bx) \ (\neg \exists x(Ax \land Bx))$$

$$egin{aligned} orall oldsymbol{x}(oldsymbol{A}oldsymbol{x} 
ightarrow oldsymbol{B}oldsymbol{x}) \ \left( 
eg \exists oldsymbol{x}(oldsymbol{A}oldsymbol{x} \wedge oldsymbol{B}oldsymbol{x}) 
ight) \end{aligned}$$

 $\exists x (Ax \wedge Bx)$ 

All A are B

Some A are B

All A are not B (No A is B)

Some A are not B (Not all A are B)

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$$\forall x (Ax \rightarrow Bx)$$

$$\exists x (Ax \wedge Bx)$$

$$\forall x (Ax \rightarrow \neg Bx) \ (\neg \exists x (Ax \land Bx))$$
  
 $\exists x (Ax \land \neg Bx) \ (\neg \forall x (Ax \rightarrow Bx))$ 

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$$\exists x (Ax \wedge Bx)$$

$$\exists oldsymbol{x} (oldsymbol{A} oldsymbol{x} \wedge 
eg oldsymbol{B} oldsymbol{x}) \ ig( 
eg oldsymbol{v} oldsymbol{x} (oldsymbol{A} oldsymbol{x} o oldsymbol{B} oldsymbol{x}) ig)$$

All A are B $\forall x (Ax \rightarrow Bx)$ Some A are B $\exists x (Ax \wedge Bx)$  $\forall x (Ax \to \neg Bx) \ (\neg \exists x (Ax \land Bx))$ All A are not B (No A is B)  $\exists x (Ax \land \neg Bx) (\neg \forall x (Ax \rightarrow Bx))$ Some A are not B (Not all A are B)  $\forall x (Ax \rightarrow Bx)$  $\exists x (Ax \wedge Bx)$ 

• We can talk about **relations** between two or more objects:

John sees Mary:

Mary sees John:

John gives Mary the book:

• We can talk about **relations** between two or more objects:

John sees Mary: Sjm
Mary sees John:
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• We can express more complicated quantifier petterns:

Everyone sees someone:

Someone sees everyone:

Everyone is seen by someone:

Someone is seen by everyone:

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Someone sees everyone:	$\exists m{x} \forall m{y}(m{S}m{x}m{y})$
Everyone is seen by someone:	$\forall oldsymbol{x} \exists oldsymbol{y}(oldsymbol{S} oldsymbol{y} oldsymbol{x})$
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Someone is seen by everyone:	$\exists m{x} orall m{y}(m{S}m{y}m{x})$

## Examples

 $\begin{array}{l} \boldsymbol{L}\boldsymbol{x}\boldsymbol{y} - \boldsymbol{x} \text{ loves } \boldsymbol{y} \\ \boldsymbol{G}\boldsymbol{x} - \boldsymbol{x} \text{ is a girl} \\ \boldsymbol{B}\boldsymbol{x} - \boldsymbol{x} \text{ is a boy} \end{array}$ 

## Examples

Lxy - x loves y Gx - x is a girl Bx - x is a boy

Every boy loves a girl

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$$\forall oldsymbol{x} ig( oldsymbol{B} oldsymbol{x} o oldsymbol{arphi}(oldsymbol{x}) ig)$$

$$Lxy - x$$
 loves  $y$   
 $Gx - x$  is a girl  
 $Bx - x$  is a boy

Every boy loves a girl 
$$\forall \boldsymbol{x} \big( \boldsymbol{B} \boldsymbol{x} \to \underline{\boldsymbol{\varphi}(\boldsymbol{x})} \big)$$
 
$$\underline{\qquad \qquad \boldsymbol{x} \text{ loves a girl} \qquad \qquad \boldsymbol{\varphi}(\boldsymbol{x})}$$

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$$\forall x \left( Bx \to \underline{\varphi}(x) \right)$$
  
 $\underline{\qquad} x \text{ loves a girl} \qquad \exists y (Gy \land Lxy)$ 

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$$\forall oldsymbol{x} ig( oldsymbol{B} oldsymbol{x} 
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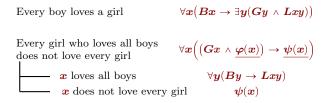
Every girl who loves all boys does not love every girl

$$orall oldsymbol{x} \Big( ig( oldsymbol{G} oldsymbol{x} \wedge \underline{oldsymbol{arphi}(oldsymbol{x})} ig) 
ightarrow \underline{oldsymbol{\psi}(oldsymbol{x})} \Big)$$

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Every boy loves a girl	$\forall x ig( Bx  ightarrow \exists y (Gy \wedge Lxy) ig)$
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$\boldsymbol{x}$ loves all boys	$oldsymbol{arphi}(oldsymbol{x})$
<b>x</b> does not love every	girl $oldsymbol{\psi}(oldsymbol{x})$

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ightarrow oldsymbol{Lxy} ig) ig) 
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Every boy loves a girl  $\forall \boldsymbol{x} \big( \boldsymbol{B} \boldsymbol{x} \to \exists \boldsymbol{y} (\boldsymbol{G} \boldsymbol{y} \wedge \boldsymbol{L} \boldsymbol{x} \boldsymbol{y}) \big)$  Every girl who loves all boys does not love every girl  $\forall \boldsymbol{x} \Big( \big( \boldsymbol{G} \boldsymbol{x} \wedge \forall \boldsymbol{y} (\boldsymbol{B} \boldsymbol{y} \to \boldsymbol{L} \boldsymbol{x} \boldsymbol{y}) \big) \to \\ \neg \forall \boldsymbol{x} (\boldsymbol{G} \boldsymbol{z} \to \boldsymbol{L} \boldsymbol{x} \boldsymbol{z}) \Big)$ 

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$$\forall \boldsymbol{x} \Big( \big( \boldsymbol{G} \boldsymbol{x} \wedge \forall \boldsymbol{y} (\boldsymbol{B} \boldsymbol{y} \to \boldsymbol{L} \boldsymbol{x} \boldsymbol{y}) \big) \to$$
 does not love every girl 
$$\neg \forall \boldsymbol{z} (\boldsymbol{G} \boldsymbol{z} \to \boldsymbol{L} \boldsymbol{x} \boldsymbol{z}) \Big)$$

•  $\neg \forall x \varphi x$  is equivalent to  $\exists x \neg \varphi x$ 

•  $\neg \forall x \varphi x$  is equivalent to  $\exists x \neg \varphi x$ •  $\neg \exists x \varphi x$  is equivalent to  $\forall x \neg \varphi x$ 

•	$\neg  \forall x  \varphi x$	is equivalent to	$\exists x egarphi x$
•	$\neg\exists x\varphi x$	is equivalent to	$orall oldsymbol{x}  eg oldsymbol{arphi} oldsymbol{x}$
•	orall x arphi x	is equivalent to	$\neg \exists x  \neg \varphi x$

•	$ eg \forall x  oldsymbol{arphi} x$	is equivalent to	$\exists x  eg arphi x$
•	$\neg\exists \boldsymbol{x}\boldsymbol{\varphi}\boldsymbol{x}$	is equivalent to	$orall oldsymbol{x}  eg oldsymbol{arphi} oldsymbol{x}$
•	$orall oldsymbol{x} oldsymbol{arphi} oldsymbol{x}$	is equivalent to	$ eg \exists x   eg \varphi x$
•	$\exists x arphi x$	is equivalent to	$ eg \forall x   eg \varphi x$

•	$ eg orall oldsymbol{x} oldsymbol{arphi} oldsymbol{x}$	is equivalent to	$\exists x\neg\varphi x$
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•	$orall oldsymbol{x} oldsymbol{arphi} oldsymbol{x}$	is equivalent to	$\neg \exists x  \neg \varphi x$
•	$\exists oldsymbol{x} oldsymbol{arphi} oldsymbol{x}$	is equivalent to	$\neg \forall \boldsymbol{x}  \neg \boldsymbol{\varphi} \boldsymbol{x}$
•	$ eg \forall x \left( arphi x  ightarrow \psi x  ight)$	is equivalent to	$\exists x \lnot (arphi x  o \psi x)$

•	$ eg \forall x  arphi x$	is equivalent to	$\exists x\neg\varphi x$
•	$\neg \exists \boldsymbol{x}  \boldsymbol{\varphi} \boldsymbol{x}$	is equivalent to	$\forall \boldsymbol{x}\neg\boldsymbol{\varphi}\boldsymbol{x}$
•	$orall oldsymbol{x} oldsymbol{arphi} oldsymbol{x}$	is equivalent to	$\neg \exists \boldsymbol{x}  \neg \boldsymbol{\varphi} \boldsymbol{x}$
•	$\exists x arphi x$	is equivalent to	$\neg \forall \boldsymbol{x}  \neg \boldsymbol{\varphi} \boldsymbol{x}$
•	$ eg \forall x \left( arphi x  ightarrow \psi x  ight)$	is equivalent to	$\exists x  \neg (\varphi x  o \psi x)$
		is equivalent to	$\exists oldsymbol{x} \left(oldsymbol{arphi} oldsymbol{x} \wedge  eg oldsymbol{\psi} ight)$

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is equivalent to
                                                                                         \exists x \, \neg \varphi x
                \neg \forall x \, \varphi x

eg \exists x \, arphi x
                                           is equivalent to
                                                                                        \forall x \neg \varphi x
                orall oldsymbol{x} oldsymbol{arphi} oldsymbol{x}
                                            is equivalent to

eg \exists x \, 
eg \varphi x
                  \exists x \, \varphi x
                                            is equivalent to
                                                                                      \neg orall x \, 
eg \varphi x
     \neg orall x \left( arphi x 
ightarrow \psi x 
ight)
                                                                                 \exists x \neg (\varphi x \rightarrow \psi x)
                                            is equivalent to
                                            is equivalent to \exists x (\varphi x \land \neg \psi)
• \forall x (\varphi x \rightarrow \neg \psi x)
                                            is equivalent to \neg \exists x \neg (\varphi x \rightarrow \neg \psi x)
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is equivalent to
                 \neg \forall x \varphi x
                                                                                               \exists x\, 
eg \varphi x

eg \exists x \, \varphi x
                                               is equivalent to
                                                                                               \forall x \neg \varphi x
                  orall oldsymbol{x} oldsymbol{arphi} oldsymbol{x}
                                               is equivalent to

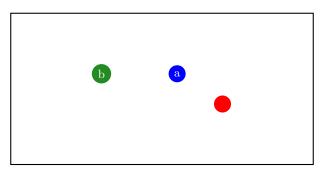
eg \exists x \, 
eg \varphi x
                   \exists x \, \varphi x
                                               is equivalent to
                                                                                             \neg orall x \, 
eg \varphi x
      \neg \forall x (\varphi x \rightarrow \psi x)
                                                                                      \exists x \neg (\varphi x \rightarrow \psi x)
                                               is equivalent to
                                               is equivalent to
                                                                                        \exists x \ (\varphi x \land \neg \psi)
• \forall x (\varphi x \rightarrow \neg \psi x)
                                               is equivalent to \neg \exists x \neg (\varphi x \rightarrow \neg \psi x)
                                                                                     \neg \exists x \ (\varphi x \wedge \psi x)
                                                is equivalent to
```

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•	$orall x \left( arphi x  ightarrow  eg \psi x  ight)$	is equivalent to	$ eg\exists x  eg(arphi x  o  eg\psi x)$
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		is equivalent to	$ eg \exists x \ (oldsymbol{arphi} x \wedge oldsymbol{\psi} x)$
•	$orall oldsymbol{x} \left(oldsymbol{arphi} oldsymbol{x} \wedge oldsymbol{\psi} oldsymbol{x} ight)$	is equivalent to	$\forall x\varphi x\wedge\forall x\psi x$
•	$\exists x  (oldsymbol{arphi} x ee oldsymbol{\psi} x)$	is equivalent to	$\exists x \varphi x \vee \exists x \psi x$

### Evaluating predicate logic formulas (1)

Colors (Red, Green, Blue, Purple) and shapes (Square, Circle).

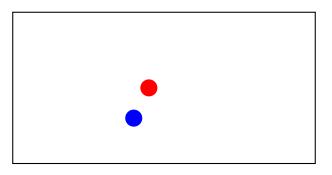


- $\bullet Ba$
- $\exists xSx \lor Cb$
- $Ra \rightarrow Sb$

- $Ba \wedge Gb$
- $\bullet \neg Sa$
- $ullet Ra 
  ightarrow \exists x Sx$

## Evaluating predicate logic formulas (1)

Colors (Red, Green, Blue, Purple) and shapes (Square, Circle).

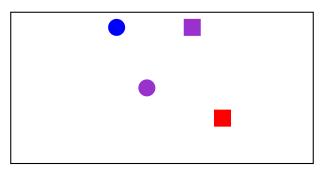


- $\bullet \exists x R x$
- $\exists x (Gx \land Cx)$

- $\bullet \neg \forall x \neg Rx$
- $\bullet \ \forall x (Rx \to Cx) \\ \\ \bullet \ \forall x (Rx \land Cx)$ 
  - $\bullet \; \exists x (Gx \to Cx)$

## Evaluating predicate logic formulas (1)

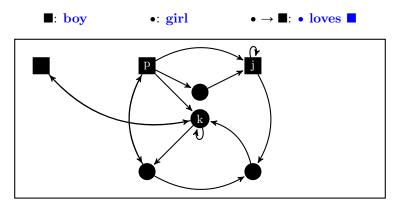
Colors (Red, Green, Blue, Purple) and shapes (Square, Circle).



- $\forall x (Cx \vee Sx)$   $\forall x Cx \vee \forall x Sx$
- ullet  $\exists xGx \lor \exists xCx$
- $\exists x (Rx \land Cx)$   $\exists x Rx \land \exists x Cx$ 

  - $\bullet \ \exists x (Gx \lor Cx)$

# Evaluating predicate logic formulas (2)



- $Lik \rightarrow Lki$
- $\neg (Ljk \wedge Lkj)$
- $\forall x (Bx \rightarrow Lxk)$
- $\forall x((Bx \vee Gx) \rightarrow \neg Lxp)$   $\exists x(Gx \wedge Lpx \wedge Lxj)$
- $Lik \wedge Lki$
- $ullet (Ljk \wedge Lpk) 
  ightarrow (
  eg Lpj \wedge 
  eg Lkj) \ ullet (Gx 
  ightarrow Lxx)$

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- **2** A **formula** is built via the following rules.
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$$Pt_1 \cdots t_n$$

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• If  $\varphi$  and  $\psi$  are formulas, then the following are formulas:

$$\neg \varphi$$
,  $\varphi \wedge \psi$ ,  $\varphi \vee \psi$ ,  $\varphi \rightarrow \psi$ ,  $\varphi \leftrightarrow \psi$ 

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• If  $\varphi$  and  $\psi$  are formulas, then the following are formulas:

$$eg arphi, \quad arphi \wedge \psi, \quad arphi \vee \psi, \quad arphi 
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• If  $\varphi$  is a formula and x is a variable, the following are formulas:

$$\forall x \varphi, \; \exists x \varphi$$



### Examples of formulas

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 $\bullet$   $Lxx \land \neg Lmx$ 

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- $\bullet \exists x Ljx$

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- $\bullet \ \forall x \big( \boldsymbol{B} \boldsymbol{x} \to \exists \boldsymbol{y} (\boldsymbol{G} \boldsymbol{y} \wedge \boldsymbol{L} \boldsymbol{x} \boldsymbol{y}) \big)$

- $\bullet$   $Lxx \land \neg Lmx$
- $\bullet \exists x Ljx$
- $\bullet \ \forall x \ Ljx$
- $\bullet \ \forall x \big( Bx \to \exists y (Gy \land Lxy) \big)$
- $\bullet \neg \exists x (Gx \land \forall y (\neg Lxy))$

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- $\bullet \neg \exists x (Gx \land \forall y (\neg Lxy))$
- $\bullet \ \forall x \big(\exists y (Lxy) \to \exists z (Lzx)\big)$
- $\bullet \exists y \forall x (Lyx)$

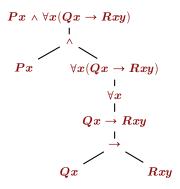
• Scope of a quantifier. In a formula of the form  $\forall x \varphi \ (\exists x \varphi)$ , the subformula  $\varphi$  is said to be **the scope** of the quantifier  $\forall \ (\exists)$ .

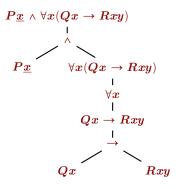
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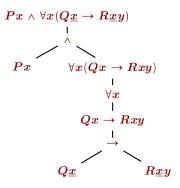
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- Free variable. An occurrence of a variable x is free in a formula  $\varphi$  if no quantifier in  $\varphi$  binds it.

$$Px \, \land \, \forall x (Qx \to Rxy)$$

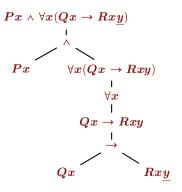




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- These occurrences of x are bound (by  $\forall x$ ).



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- This occurrence of y is not bound by any quantifier (is free).

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- Open formula. A formula is open if it is not open, that is, if it contains at least one free occurrence of a variable.

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 $(oldsymbol{s})_{oldsymbol{t}}^{oldsymbol{y}}$ 

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$$(s)_t^y$$

• Formally,

For a **constant**: 
$$(c)_t^y := c$$
For a **variable**: 
$$\begin{cases} (x)_t^y := x & \text{for } x \text{ different from } y \\ (y)_t^y := t \end{cases}$$

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Examples:

$$egin{aligned} (oldsymbol{a})_{oldsymbol{c}}^{oldsymbol{x}} &:= oldsymbol{a} \ (oldsymbol{x})_{oldsymbol{u}}^{oldsymbol{x}} &:= oldsymbol{x} \ (oldsymbol{z})_{oldsymbol{u}}^{oldsymbol{z}} &:= oldsymbol{y} \end{aligned}$$

• Substitution inside a formula. Replacing the free occurrences of the variable y for the term t inside the formula  $\varphi$  produces the formula denoted by

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• Formally,

$$(\mathbf{P}t_{1}\cdots t_{n})_{t}^{y} := \mathbf{P}(t_{1})_{t}^{y}\cdots (t_{n})_{t}^{y}$$

$$(\neg\varphi)_{t}^{y} := \neg(\varphi)_{t}^{y}$$

$$(\varphi \wedge \psi)_{t}^{y} := (\varphi)_{t}^{y} \wedge (\psi)_{t}^{y}$$

$$(\varphi \vee \psi)_{t}^{y} := (\varphi)_{t}^{y} \vee (\psi)_{t}^{y}$$

$$(\varphi \to \psi)_{t}^{y} := (\varphi)_{t}^{y} \to (\psi)_{t}^{y}$$

$$(\varphi \leftrightarrow \psi)_{t}^{y} := (\varphi)_{t}^{y} \leftrightarrow (\psi)_{t}^{y}$$

$$egin{cases} (orall xarphi)_t^y &:= orall x(arphi)_t^y \ (orall yarphi)_t^y &:= orall yarphi \ (\exists xarphi)_t^y &:= \exists x(arphi)_t^y \ (\exists uarphi)_t^y &:= \exists uarphi \end{cases}$$

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For a constant a, sometimes we will abbreviate object I(a) as  $a_I$ . For a predicate P, sometimes we will abbreviate relation I(P) as  $P_I$ .

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The variable assignment  $g_{[\boldsymbol{x}:=\boldsymbol{d}]}$  differs from g only in the value of  $\boldsymbol{x}$ , given now by d.

## Assigning values to terms

Given a model  $M = \langle D, I, g \rangle$ , the value of every term t,  $[t]_g^I$ , is defined as follows.

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For a variable 
$$x$$
:  $[x]_q^I := g(x)$ 

$$\langle D, I, g \rangle \models Pt_1 \cdots t_n \text{ iff } (\llbracket t_1 \rrbracket_q^I, \dots, \llbracket t_n \rrbracket_q^I) \in I(P)$$

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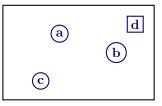
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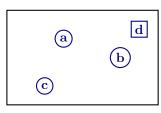
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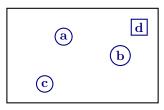
Shapes ( $\boldsymbol{S}$ quare,  $\boldsymbol{C}$ ircle).





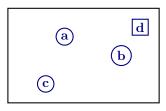
$$D := \{ (a), (b), (c), (d) \}$$

Shapes (Square, Circle).

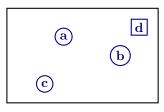


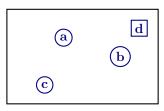
$$egin{aligned} D &:= \{ @, \ @, \ @, \ d \ \} \ & I(a) &:= \ @ \ & I(b) &:= \ @ \ & I(c) &:= \ &$$

 $I(d) := \boxed{\mathbf{d}}$ 

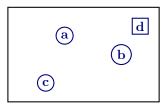


$$\begin{split} &D := \{ \textcircled{a}, \textcircled{b}, \textcircled{c}, \textcircled{d} \} \\ &I(\textbf{a}) := \textcircled{a} \qquad I(\textbf{S}) := \{ \textcircled{d} \} \\ &I(\textbf{b}) := \textcircled{b} \qquad I(\textbf{C}) := \{ \textcircled{a}, \textcircled{b}, \textcircled{c} \} \\ &I(\textbf{c}) := \textcircled{c} \\ &I(\textbf{d}) := \textcircled{d} \end{split}$$

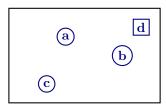




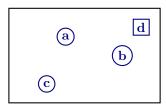
$$egin{aligned} \langle D,I,g
angle &\models Ca & ext{iff} \ \langle D,I,g
angle &\models Sx & ext{iff} \ \langle D,I,g
angle &\models \exists xSx & ext{iff} \end{aligned}$$



$$\begin{split} \langle D, I, g \rangle &\models Ca & \text{iff} & \llbracket a \rrbracket_g^I \in I(C) \\ \langle D, I, g \rangle &\models Sx & \text{iff} \\ \langle D, I, g \rangle &\models \exists x Sx & \text{iff} \end{split}$$

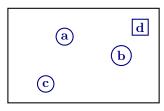


$$\langle D, I, g \rangle \models Ca$$
 iff  $I(a) \in I(C)$   
 $\langle D, I, g \rangle \models Sx$  iff  
 $\langle D, I, g \rangle \models \exists xSx$  iff



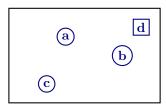
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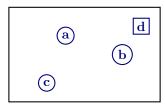
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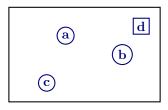
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$$\langle D, I, g \rangle \models Ca$$
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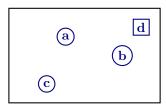
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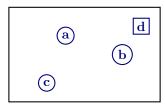
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angle &\models &\exists \pmb{xSx} & & & & & & & & & & & \\ \end{aligned}$$



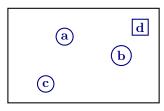
$$egin{aligned} D &:= \{ @, \ @, \ @, \ d \} \ & I(m{a}) &:= \ @ & I(m{S}) &:= \{ \ d \} \ & I(m{b}) &:= \ @ & I(m{C}) &:= \{ @, \ @, \ @, \ e \} \ & I(m{c}) &:= \ @ & g(m{x}) &:= \ @ & I(m{d}) &:= \ & g(m{y}) &:= \ @ &$$

$$\langle D, I, g \rangle \models Ca$$
 iff  $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \checkmark$   
 $\langle D, I, g \rangle \models Sx$  iff  $\textcircled{b} \in I(S)$   
 $\langle D, I, g \rangle \models \exists xSx$  iff



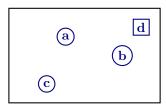
$$egin{aligned} D &:= \{ @, @, @, @, @ \} \ & I(a) &:= @ & I(S) &:= \{ @, \} \ & I(b) &:= & B & I(C) &:= \{ @, @, @, @ \} \ & I(c) &:= & B & g(x) &:= & B \ & I(d) &:= & G & g(y) &:= & B \end{aligned}$$

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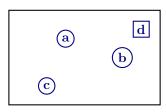


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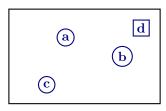
$$\langle D, I, g \rangle \models \exists x S x \quad \text{iff} \quad \text{there is a } o \in D \text{ such that } \langle D, I, g_{[x:=o]} \rangle \models S x$$

Shapes (Square, Circle).



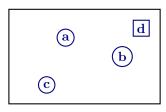
$$egin{aligned} D &:= \{ @, \ @, \ @, \ @ \ \end{bmatrix} \ & I(m{a}) &:= \ @ \ & I(m{S}) &:= \{ \ @, \ @, \ @ \ \end{bmatrix} \ & I(m{b}) &:= \ @ \ & I(m{C}) &:= \{ @, \ @, \ @, \ @ \ \end{bmatrix} \ & I(m{c}) &:= \ @ \ & g(m{x}) &:= \ @ \ & I(m{d}) &:= \ @ \ & g(m{y}) &:= \ @ \ & \end{bmatrix}$$

$$\begin{split} \langle D, I, g \rangle &\models Ca & \text{iff} & \textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \checkmark \\ \langle D, I, g \rangle &\models Sx & \text{iff} & \textcircled{b} \in \{\textcircled{d}\} \checkmark \\ \langle D, I, g \rangle &\models \exists xSx & \text{iff} & \text{there is a } o \in D \text{ such that } \langle D, I, g_{[x:=o]} \rangle \models Sx \\ & \langle D, I, g_{[x:=a]} \rangle \models Sx \end{split}$$



$$egin{aligned} D &:= \{ @, \&b, \&c, \&d \} \ &I(m{a}) &:= \&a & I(m{S}) &:= \&d \} \ &I(m{b}) &:= \&b & I(m{C}) &:= \&a, \&b, \&c \} \ &I(m{c}) &:= \&c & g(m{x}) &:= \&b \ &I(m{d}) &:= \&d & g(m{y}) &:= \&a \end{aligned}$$

$$\begin{split} \langle D, I, g \rangle &\models Ca & \text{iff} & \textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \checkmark \\ \langle D, I, g \rangle &\models Sx & \text{iff} & \textcircled{b} \in \{\textcircled{d}\} \checkmark \\ \langle D, I, g \rangle &\models \exists xSx & \text{iff} & \text{there is a } o \in D \text{ such that } \langle D, I, g_{[x:=o]} \rangle &\models Sx \\ & & \|x\|_{g_{[x:=o]}}^{I} \in I(S) \end{split}$$

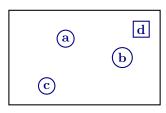


$$egin{aligned} D &:= \{ @, \&b, \&c, \&d \} \ I(a) &:= @ & I(S) := \{\&d\} \ I(b) &:= \&b & I(C) := \{\&a, \&b, \&c\} \ I(c) &:= \&c & g(x) := \&b \ I(d) &:= \&d & g(y) := \&a \end{aligned}$$

$$\begin{split} &\langle D, I, g \rangle \models \pmb{Ca} & \text{iff} & \textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \checkmark \\ &\langle D, I, g \rangle \models \pmb{Sx} & \text{iff} & \textcircled{b} \in \{\textcircled{d}\} \checkmark \\ &\langle D, I, g \rangle \models \exists \pmb{xSx} & \text{iff} & \text{there is a } o \in D \text{ such that } \langle D, I, g_{[\pmb{x}:=o]} \rangle \models \pmb{Sx} \end{split}$$

$$g_{[x:=@]}(x) \in I(S)$$

Shapes (Square, Circle).

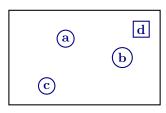


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$$\begin{split} \langle D, I, g \rangle &\models Ca & \text{iff} & \textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \checkmark \\ \langle D, I, g \rangle &\models Sx & \text{iff} & \textcircled{b} \in \{\textcircled{d}\} \checkmark \\ \langle D, I, g \rangle &\models \exists x Sx & \text{iff} & \text{there is a } o \in D \text{ such that } \langle D, I, g_{[x:=o]} \rangle &\models Sx \end{split}$$

 $\mathbf{a} \in \boldsymbol{I}(\boldsymbol{S})$ 

Shapes (Square, Circle).

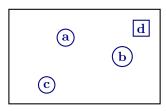


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$$egin{array}{ll} \langle D,I,g
angle \models \pmb{Ca} & ext{iff} & ext{ a) } \in \{\mbox{a)}, \mbox{ c)} \ \checkmark \ \\ \langle \pmb{D},\pmb{I},g
angle \models \pmb{Sx} & ext{iff} & ext{ b) } \in \{\mbox{d]} \ \nearrow \ \\ \langle \pmb{D},\pmb{I},g
angle \models \exists \pmb{xSx} & ext{iff} & ext{there is a } \pmb{o} \in \pmb{D} \text{ such that } \langle \pmb{D},\pmb{I},g_{[\pmb{x}:=\pmb{o}]}
angle \models \pmb{Sx} \ \end{array}$$

(a) ∈ {[d]}

Shapes (Square, Circle).

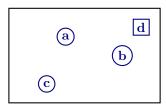


$$\langle D, I, g \rangle \models Ca$$
 iff  $(a) \in \{(a), (b), (c)\}$   $\checkmark$   $\langle D, I, g \rangle \models Sx$  iff  $(b) \in \{(d)\}$   $\nearrow$   $\langle D, I, g \rangle \models \exists x Sx$  iff there is a  $g \in D$  such

 $\langle D, I, g \rangle \models \exists x S x$  iff there is a  $o \in D$  such that  $\langle D, I, g_{[x:=o]} \rangle \models S x$ 



Shapes (Square, Circle).

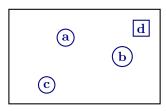


$$\begin{split} & D := \{ \texttt{a}, \, \texttt{b}, \, \texttt{c}, \, \texttt{d} \} \\ & I(\pmb{a}) := \, \texttt{a} \qquad I(\pmb{S}) := \{ \texttt{d} \} \\ & I(\pmb{b}) := \, \texttt{b} \qquad I(\pmb{C}) := \, \{ \texttt{a}, \, \texttt{b}, \, \texttt{c} \} \\ & I(\pmb{c}) := \, \texttt{c} \qquad g(\pmb{x}) := \, \texttt{b} \\ & I(\pmb{d}) := \, \texttt{d} \qquad g(\pmb{y}) := \, \texttt{a} \end{split}$$

$$\langle D, I, g \rangle \models Ca$$
 iff  $\textcircled{a} \in \{\textcircled{a}, \textcircled{b}, \textcircled{c}\} \checkmark$   
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 $\langle D, I, g \rangle \models \exists x S x \quad \text{iff} \quad \text{there is a } o \in D \text{ such that } \langle D, I, g_{[x:=o]} \rangle \models S x$ 

. . .

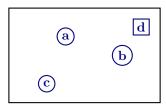


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$$\langle oldsymbol{D}, oldsymbol{I}, oldsymbol{g}_{[oldsymbol{x}:=oldsymbol{d}]} 
angle \models oldsymbol{S}oldsymbol{x}$$

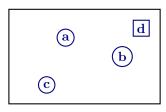
Shapes (Square, Circle).



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Shapes (Square, Circle).

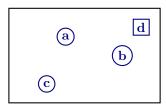


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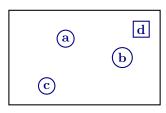
$$g_{[x:=d]}(x) \in I(S)$$

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Shapes (Square, Circle).



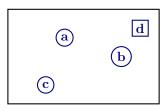
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$$\begin{split} &\langle D, I, g \rangle \models Ca & \text{iff} & \texttt{ (a)} \in \{\texttt{ (a)}, \texttt{ (b)}, \texttt{ (c)} \} \checkmark \\ &\langle D, I, g \rangle \models Sx & \text{iff} & \texttt{ (b)} \in \{\texttt{ (d)} \} \checkmark \\ &\langle D, I, g \rangle \models \exists xSx & \text{iff} & \text{there is a } o \in D \text{ such that } \langle D, I, g_{[x:=o]} \rangle \models Sx \end{split}$$

 $\mathbf{d} \in \{\mathbf{d}\}$ 

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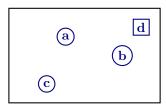
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Shapes (Square, Circle).



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 $\langle D, 1, g \rangle = \exists x \cup x$  in there is a  $u \in D$  such that  $\langle D, 1, g |_{x := 0} \rangle = \exists x \cup x$ 

# Definitions (1)

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- An inference  $\varphi_1, \ldots, \varphi_n/\psi$  is (logically) valid if, for every model M for which we have  $M \models \varphi_1, \ldots, M \models \varphi_n$ , we also have  $M \models \psi$ . In such case we will write  $\varphi_1, \ldots, \varphi_n \models \psi$ .

# Definitions (2)

In particular, for two formulas  $\varphi$  and  $\psi$ ,

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•  $\psi$  (logically) follows from  $\varphi$  if

$$arphi \models \psi$$

•  $\psi$  is (logically) equivalent to  $\varphi$  if

$$\varphi \models \psi$$
 and  $\psi \models \varphi$ 

The valid formulas of predicate logic can be derived from the following principles:

All propositional tautologies.

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- $varproption \forall x \varphi \to (\varphi)_t^x$ , provided that no variable in t occurs bounded in  $\varphi$ .

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The valid formulas of predicate logic can be derived from the following principles:

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- **6** Modus ponens (MP): from  $\varphi$  and  $\varphi \to \psi$ , infer  $\varphi$ .
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A formula that can be derived by following these principles in a *finite* number of steps is called a **theorem**.



1.

$$\forall x \neg \varphi \to (\neg \varphi)_t^x$$

Axiom 2



- 1.  $\forall x \neg \varphi \to (\neg \varphi)_t^x$
- $\forall x \neg \varphi \rightarrow \neg (\varphi)_t^x$

Axiom 2

 ${\bf Definition\ of\ substitution}$ 

1. 
$$\forall x \neg \varphi \to (\neg \varphi)_t^x$$

2. 
$$\forall x \neg \varphi \rightarrow \neg (\varphi)$$

2. 
$$\forall x \neg \varphi \rightarrow \neg (\varphi)_t^x$$
3. 
$$(\forall x \neg \varphi \rightarrow \neg (\varphi)_t^x) \rightarrow ((\varphi)_t^x \rightarrow \neg \forall x \neg \varphi)$$

Axiom 2

Definition of substitution

Propositional tautology

$$\begin{aligned} 1. & \forall x \neg \varphi \rightarrow (\neg \varphi)_t^x \\ 2. & \forall x \neg \varphi \rightarrow \neg (\varphi)_t^x \\ 3. & \left( \forall x \neg \varphi \rightarrow \neg (\varphi)_t^x \right) \rightarrow \left( (\varphi)_t^x \rightarrow \neg \forall x \neg \varphi \right) \\ 4. & (\varphi)_t^x \rightarrow \neg \forall x \neg \varphi \end{aligned}$$

Axiom 2
Definition of substitution
Propositional tautology

MP from steps 2 and 3

1. 
$$\forall x \neg \varphi \rightarrow (\neg \varphi)_t^x$$
2. 
$$\forall x \neg \varphi \rightarrow \neg (\varphi)_t^x$$
3. 
$$\left(\forall x \neg \varphi \rightarrow \neg (\varphi)_t^x\right) \rightarrow \left((\varphi)_t^x \rightarrow \neg \forall x \neg \varphi\right)$$
4. 
$$(\varphi)_t^x \rightarrow \neg \forall x \neg \varphi$$
5. 
$$(\varphi)_t^x \rightarrow \exists x \varphi$$

Definition of substitution Propositional tautology MP from steps 2 and 3 Axiom 5

Axiom 2

$$\begin{aligned} 1. & \forall x \neg \varphi \rightarrow (\neg \varphi)_t^x \\ 2. & \forall x \neg \varphi \rightarrow \neg (\varphi)_t^x \\ 3. & \left( \forall x \neg \varphi \rightarrow \neg (\varphi)_t^x \right) \rightarrow \left( (\varphi)_t^x \rightarrow \neg \forall x \neg \varphi \right) \\ 4. & (\varphi)_t^x \rightarrow \neg \forall x \neg \varphi \\ 5. & (\varphi)_t^x \rightarrow \exists x \varphi \end{aligned}$$

Hence,  $(\varphi)_{t}^{x} \to \exists x \varphi$  is a theorem.

Axiom 2

Definition of substitution

 ${\bf Propositional\ tautology}$ 

MP from steps 2 and 3

Axiom 5

The given proof system has two properties:

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• Soundness: every derivable theorem is a valid formula.

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- Soundness: every derivable theorem is a valid formula.
- Completeness: every valid formula is a derivable theorem.

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- Example: "John loves Mary and Bill loves another girl"

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- Example: "John loves Mary and Bill loves another girl"

$$Ljm \wedge \exists x (Gx \wedge \neg (x = m) \wedge Lbx)$$

ullet  $\langle D, I, g \rangle \models t_1 = t_2$  iff  $\llbracket t_1 \rrbracket_g^I$  and  $\llbracket t_2 \rrbracket_g^I$  are the same object.



• Symbols for **functions**:

$$oldsymbol{f}, oldsymbol{g}, oldsymbol{h}, \dots$$

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• If  $t_1, \ldots, t_n$  are terms and f is a function symbol,  $f(t_1, \ldots, t_n)$  is a term.

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- $\bullet \ \ \text{The value of a function: } \llbracket f(t_1,\ldots,t_n) \rrbracket_g^I := I(f) \big( \llbracket t_1 \rrbracket_g^I,\ldots,\llbracket t_n \rrbracket_g^I \big).$
- Example: the successor function s is given by I(s)(n) := n + 1.

