Logic in Action

Chapter 6: Logic and Action

http://www.logicinaction.org/

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- *They do nothing.*

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- After *they do nothing*, everything stays the same.

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- **Converse.** Undo an executed action:

  Close the window you just opened.

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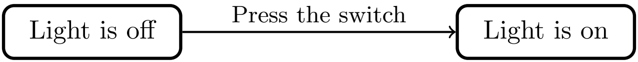

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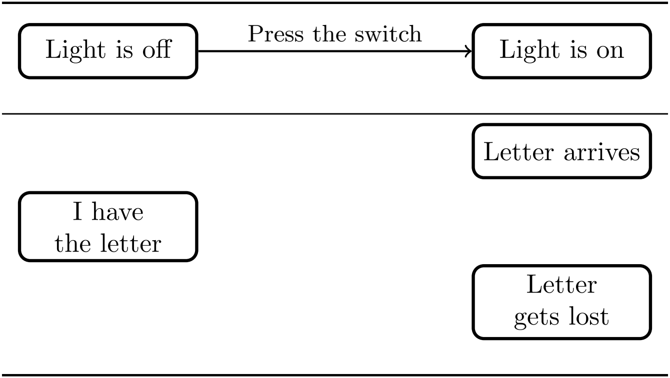

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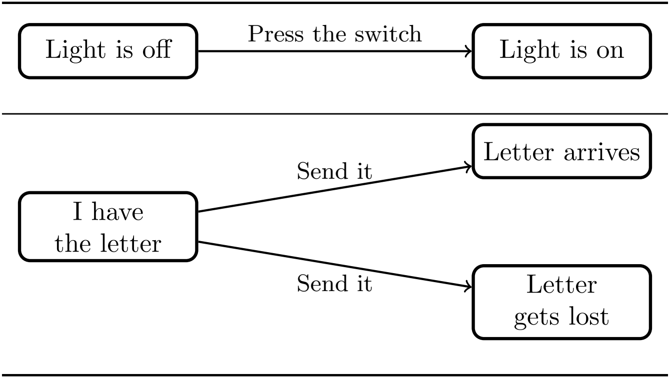

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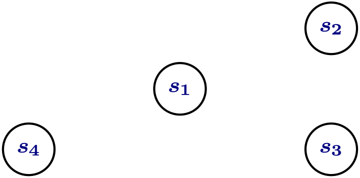

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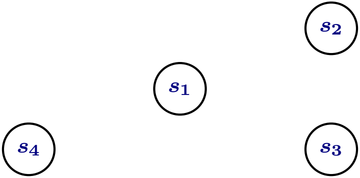

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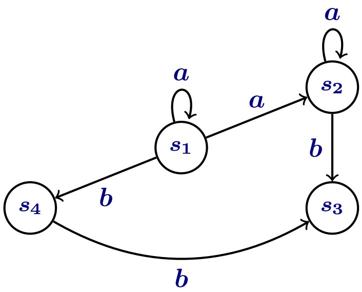

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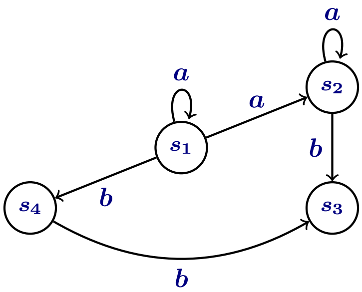

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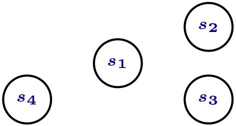

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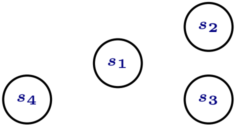

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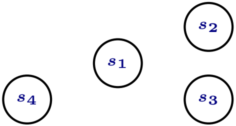

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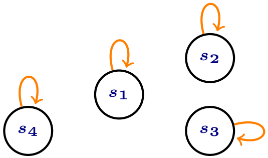

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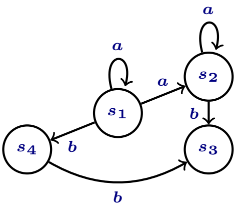

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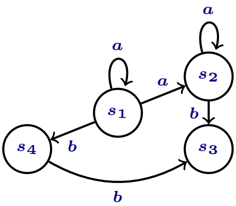

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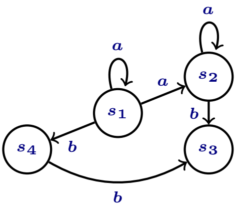

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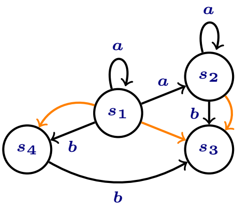

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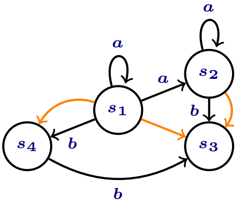

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In particular, for any relation R_a, we have

$$R_a^0 := I, \qquad R_a^1 := R_a \circ R_a^0, \qquad R_a^2 := R_a \circ R_a^1, \qquad R_a^3 := R_a \circ R_a^2,$$

and so on.

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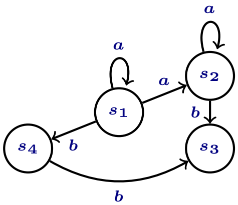

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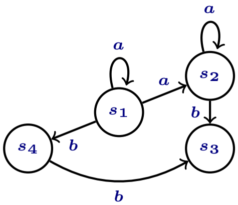

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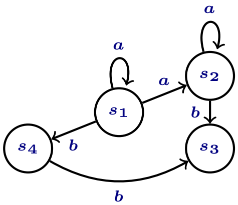

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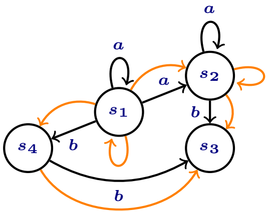

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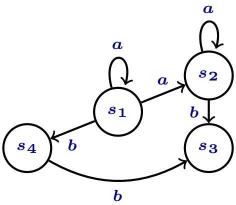

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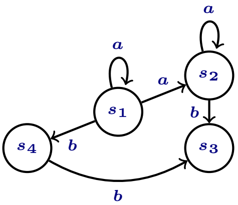

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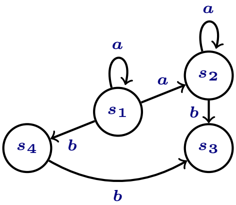

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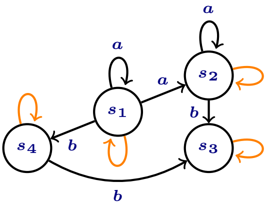

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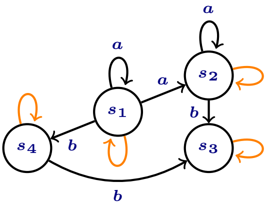

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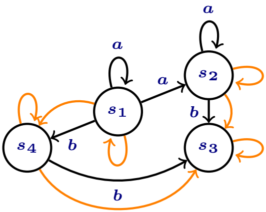

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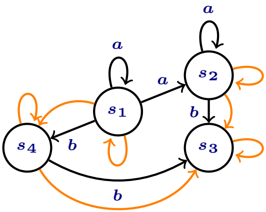

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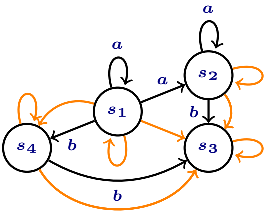

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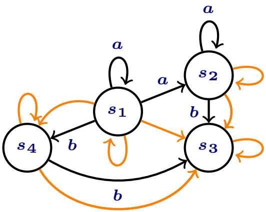

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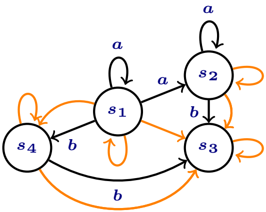

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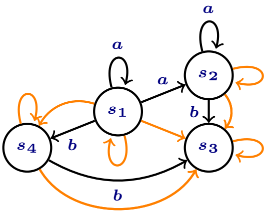

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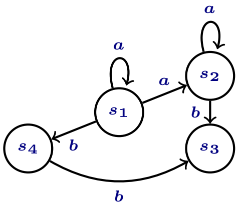

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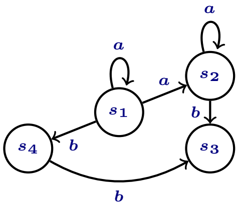

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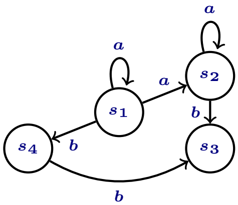

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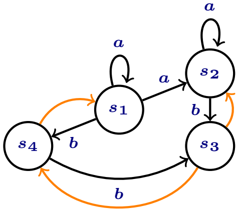

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Intuitions and abbreviations

$\alpha ; \beta$
$\alpha \cup \beta$
α^*
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$[\alpha] \varphi$	After any execution of α, φ is the case.

Some examples of formulas

$$\langle \boldsymbol{\alpha} \rangle \top$$

$$[\boldsymbol{\alpha}] \bot$$

$$\langle \boldsymbol{\alpha} \rangle \varphi \land \neg [\boldsymbol{\alpha}] \varphi$$

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$[\boldsymbol{\alpha}] \perp$	$\boldsymbol{\alpha}$ cannot be executed.
$\langle \boldsymbol{\alpha} \rangle \varphi \land \neg [\boldsymbol{\alpha}] \varphi$	$\boldsymbol{\alpha}$ can be executed it at least two different ways.

The models (1)

The structures in which we evaluate PDL formulas, **labelled transition systems (*LTS*)**, have three components:

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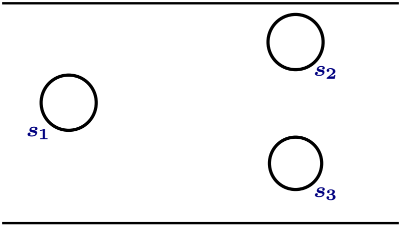

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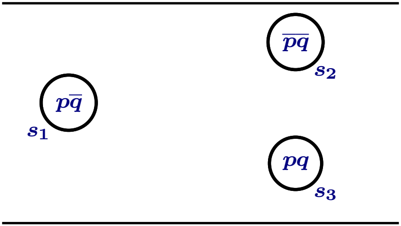

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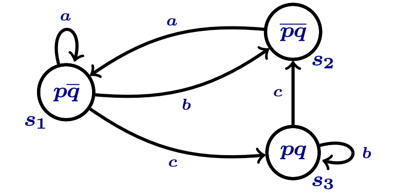

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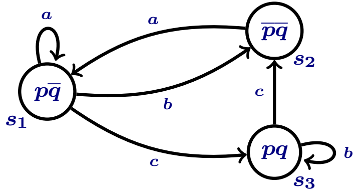
A **labelled transition system** with a designate state (the *root* state) is called a **pointed labelled transition system** or a **process graph**.



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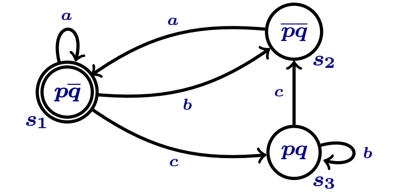

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\begin{array}{lll}
(M, s) \models p & \text{iff} & p \in V(s) \\
(M, s) \models \neg\varphi & \text{iff} & \text{it is not the case that } (M, s) \models \varphi \\
(M, s) \models \varphi \vee \psi & \text{iff} & (M, s) \models \varphi \ \text{ or } \ (M, s) \models \psi \\
\ldots & \text{iff} & \ldots \\
(M, s) \models \langle\alpha\rangle\,\varphi & \text{iff} & \text{there is a } t \in S \text{ such that } R_\alpha st \text{ and } (M, t) \models \varphi
\end{array}
$$

where the relation R_α is given, in case α is not a basic action, by

$$
\begin{aligned}
R_{\alpha;\beta} &:= R_\alpha \circ R_\beta \\
R_{\alpha\cup\beta} &:= R_\alpha \cup R_\beta \\
R_{\alpha^*} &:= (R_\alpha)^*
\end{aligned}
$$

Deciding truth-value of formulas

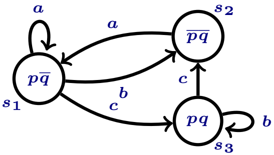
Take a pointed labelled transition system (M, s) with $M = \langle S, R_a, V\rangle$:

$$
\begin{array}{lll}
(M, s) \models p & \text{iff} & p \in V(s) \\
(M, s) \models \neg\varphi & \text{iff} & \text{it is not the case that } (M, s) \models \varphi \\
(M, s) \models \varphi \vee \psi & \text{iff} & (M, s) \models \varphi \ \text{ or } \ (M, s) \models \psi \\
\dots & \text{iff} & \dots \\
(M, s) \models \langle\alpha\rangle\,\varphi & \text{iff} & \text{there is a } t \in S \text{ such that } R_\alpha st \text{ and } (M, t) \models \varphi
\end{array}
$$

where the relation R_α is given, in case α is not a basic action, by

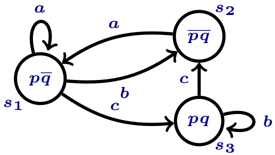
$$
\begin{aligned}
R_{\alpha;\beta} &:= R_\alpha \circ R_\beta \\
R_{\alpha\cup\beta} &:= R_\alpha \cup R_\beta \\
R_{\alpha^*} &:= (R_\alpha)^* \\
R_{?\varphi} &:= \{(s, s) \in S \times S \mid (M, s) \models \varphi\}
\end{aligned}
$$

Example: building complex relations



$$
\begin{aligned}
R_a &:= \{(s_1, s_1), (s_2, s_1)\} \\
R_b &:= \{(s_1, s_2), (s_3, s_3)\} \\
R_c &:= \{(s_1, s_3), (s_3, s_2)\}
\end{aligned}
$$

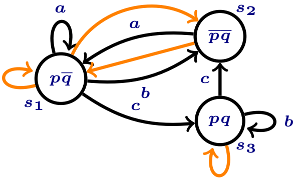
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a \cup b} =$$

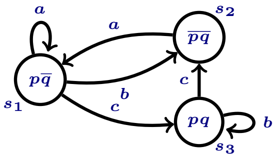
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a \cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$

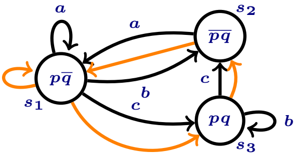
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a \cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a \cup c} =$$

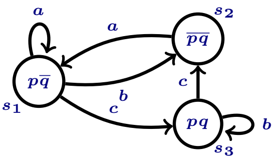
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a\cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a\cup c} = \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}$$

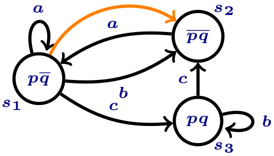
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a \cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a \cup c} = \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}$$
$$R_{c;c} =$$

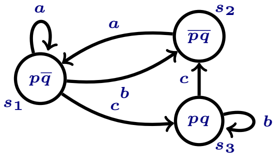
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a\cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a\cup c} = \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}$$
$$R_{c;c} = \{(s_1, s_2)\}$$

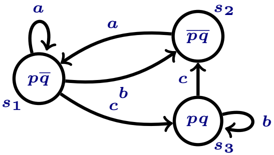
Example: building complex relations



$$
\begin{aligned}
R_a &:= \{(s_1, s_1), (s_2, s_1)\} \\
R_b &:= \{(s_1, s_2), (s_3, s_3)\} \\
R_c &:= \{(s_1, s_3), (s_3, s_2)\}
\end{aligned}
$$

$$
\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &=
\end{aligned}
$$

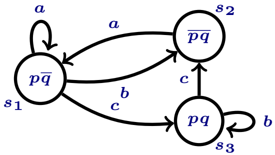
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a \cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a \cup c} = \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}$$
$$R_{c;c} = \{(s_1, s_2)\}$$
$$R_{b;b} = \{\ \}$$

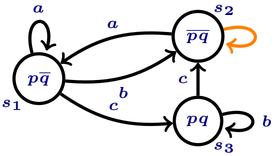
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a \cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a \cup c} = \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}$$
$$R_{c;c} = \{(s_1, s_2)\}$$
$$R_{b;b} = \{\ \}$$
$$R_{?\neg(p \vee q)} =$$

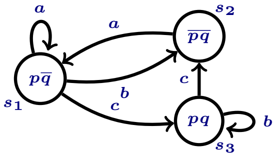
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}\\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}\\
R_{c;c} &= \{(s_1, s_2)\}\\
R_{b;b} &= \{\ \}\\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\}
\end{aligned}$$

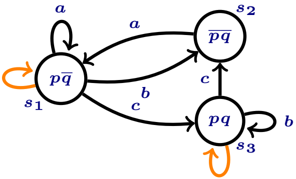
Example: building complex relations



$$
\begin{aligned}
R_a &:= \{(s_1, s_1), (s_2, s_1)\} \\
R_b &:= \{(s_1, s_2), (s_3, s_3)\} \\
R_c &:= \{(s_1, s_3), (s_3, s_2)\}
\end{aligned}
$$

$$
\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &= \{\ \} \\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\
R_{?(p\vee q)} &=
\end{aligned}
$$

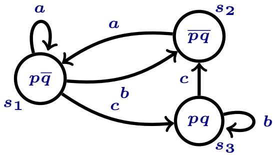
Example: building complex relations



$$\begin{aligned} R_a &:= \{(s_1, s_1), (s_2, s_1)\} \\ R_b &:= \{(s_1, s_2), (s_3, s_3)\} \\ R_c &:= \{(s_1, s_3), (s_3, s_2)\} \end{aligned}$$

$$\begin{aligned} R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\ R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\ R_{c;c} &= \{(s_1, s_2)\} \\ R_{b;b} &= \{\ \} \\ R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\ R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\} \end{aligned}$$

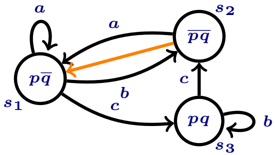
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &= \{\ \} \\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\
R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\} \\
R_{?\neg(p\vee q);a;?(p\vee q)} &=
\end{aligned}$$

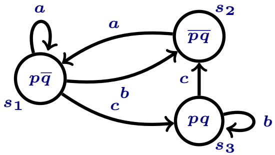
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &= \{\ \} \\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\
R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\} \\
R_{?\neg(p\vee q);a;?(p\vee q)} &= \{(s_2, s_1)\}
\end{aligned}$$

Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
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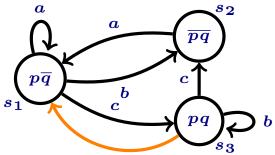
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}\\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}\\
R_{c;c} &= \{(s_1, s_2)\}\\
R_{b;b} &= \{\ \}\\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\}\\
R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\}\\
R_{?\neg(p\vee q);a;?(p\vee q)} &= \{(s_2, s_1)\}\\
R_{c;a} &=
\end{aligned}$$

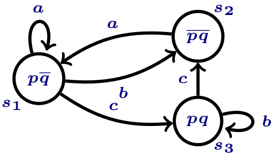
Example: building complex relations



$$
\begin{aligned}
R_a &:= \{(s_1, s_1), (s_2, s_1)\} \\
R_b &:= \{(s_1, s_2), (s_3, s_3)\} \\
R_c &:= \{(s_1, s_3), (s_3, s_2)\}
\end{aligned}
$$

$$
\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &= \{\ \} \\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\
R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\} \\
R_{?\neg(p\vee q);a;?(p\vee q)} &= \{(s_2, s_1)\} \\
R_{c;a} &= \{(s_3, s_1)\}
\end{aligned}
$$

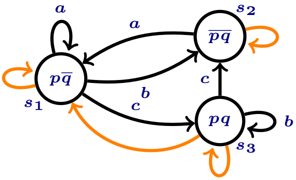
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$R_{a\cup b} = \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\}$$
$$R_{a\cup c} = \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\}$$
$$R_{c;c} = \{(s_1, s_2)\}$$
$$R_{b;b} = \{\ \}$$
$$R_{?\neg(p\vee q)} = \{(s_2, s_2)\}$$
$$R_{?(p\vee q)} = \{(s_1, s_1), (s_3, s_3)\}$$
$$R_{?\neg(p\vee q);a;?(p\vee q)} = \{(s_2, s_1)\}$$
$$R_{c;a} = \{(s_3, s_1)\}$$
$$R_{(c;a)^*} =$$

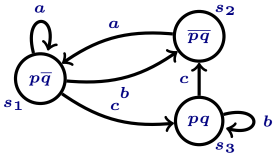
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

$$\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &= \{\ \} \\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\
R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\} \\
R_{?\neg(p\vee q);a;?(p\vee q)} &= \{(s_2, s_1)\} \\
R_{c;a} &= \{(s_3, s_1)\} \\
R_{(c;a)^*} &= \{(s_3, s_1), (s_1, s_1), (s_2, s_2), (s_3, s_3)\}
\end{aligned}$$

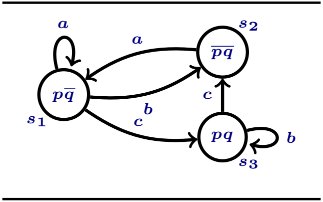
Example: building complex relations



$$R_a := \{(s_1, s_1), (s_2, s_1)\}$$
$$R_b := \{(s_1, s_2), (s_3, s_3)\}$$
$$R_c := \{(s_1, s_3), (s_3, s_2)\}$$

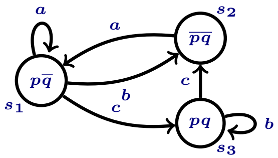
$$\begin{aligned}
R_{a\cup b} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_2), (s_3, s_3)\} \\
R_{a\cup c} &= \{(s_1, s_1), (s_2, s_1), (s_1, s_3), (s_3, s_2)\} \\
R_{c;c} &= \{(s_1, s_2)\} \\
R_{b;b} &= \{\ \} \\
R_{?\neg(p\vee q)} &= \{(s_2, s_2)\} \\
R_{?(p\vee q)} &= \{(s_1, s_1), (s_3, s_3)\} \\
R_{?\neg(p\vee q);a;?(p\vee q)} &= \{(s_2, s_1)\} \\
R_{c;a} &= \{(s_3, s_1)\} \\
R_{(c;a)^*} &= \{(s_3, s_1), (s_1, s_1), (s_2, s_2), (s_3, s_3)\}
\end{aligned}$$

Example: evaluating formulas



M

Example: evaluating formulas



M

$(M, s_1) \models \langle a \cup b \rangle\, p \wedge \neg [a \cup b]\, p$?

$(M, s_1) \models [b]\, \bot$?

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$(M, s_3) \models [(c; a)^*]\, p$?

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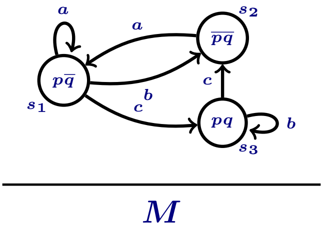
Example: evaluating formulas



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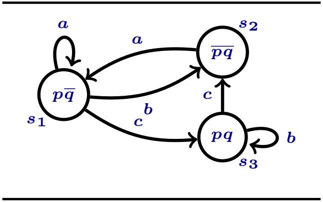

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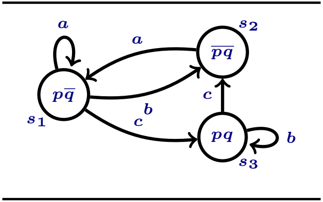

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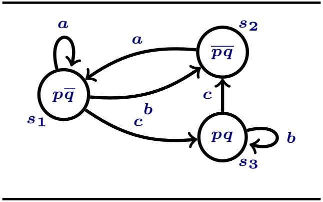

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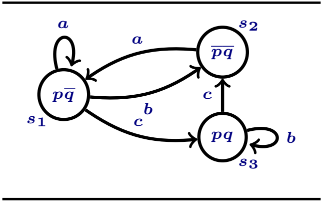

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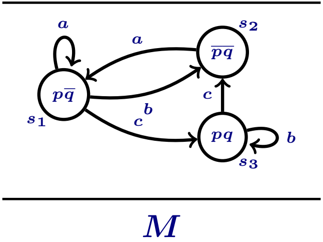

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4. **Necessitation** (Nec): from φ infer $[\alpha]\varphi$ for any action α.

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A formula that can be derived by following these principles in a *finite* number of steps is called a **theorem**.

Example

Prove that $[(\alpha \cup \beta); \gamma]\,\varphi \leftrightarrow ([\alpha; \gamma]\,\varphi \wedge [\beta; \gamma]\,\varphi)$ is valid.

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4.	$[\alpha;\gamma]\,\varphi \wedge [\beta;\gamma]\,\varphi$	Sequence from step 3

Example

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The right to left direction is similar.

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$$(?\varphi; \alpha) \cup (?\neg\varphi; \beta)$$