Logic in Action
Chapter 9: Proofs

http://www.logicinaction.org/
Systems revised so far

Issues with the **tableau** method.
Systems revised so far

Issues with the tableau method.

- It is a refutation method.
Issues with the \textit{tableau} method.

- It is a \textit{refutation} method.
- It does not follow the way humans reason.
Systems revised so far

Issues with the *tableau* method.

- It is a *refutation* method.
- It does not follow the way humans reason.

Issues with the *presented derivation systems*. 

(http://www.logicinaction.org/)
Systems revised so far

Issues with the tableau method.

- It is a refutation method.
- It does not follow the way humans reason.

Issues with the presented derivation systems.

- Proofs are not very natural (e.g., try to prove $\varphi \to \neg\neg\varphi$).
Systems revised so far

Issues with the **tableau** method.

- It is a *refutation* method.
- It does not follow the way humans reason.

Issues with the **presented derivation systems**.

- Proofs are not very natural (e.g., try to prove \( \varphi \rightarrow \neg\neg\varphi \)).
- They do not facilitate *conditional* reasoning.
The *deduction* property
The *deduction* property

\[ \Sigma, \varphi \models \psi \] if and only if \[ \Sigma \models \varphi \rightarrow \psi \]
What if we can make assumptions?

Consider a proof for \( \varphi \rightarrow \varphi \).
What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
What if we can make assumptions?

Consider a proof for $\phi \rightarrow \phi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions …
What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.

- But if we can make assumptions …

\[ 1 \quad \varphi \]
What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.

- But if we can make assumptions . . .

\[
\begin{array}{c|c}
1 & \varphi \\
2 & \varphi & \text{repetition 1}
\end{array}
\]
What if we can make assumptions?

Consider a proof for $\varphi \rightarrow \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.
- But if we can make assumptions ... 

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\varphi$</td>
</tr>
<tr>
<td>2</td>
<td>$\varphi$</td>
</tr>
<tr>
<td>3</td>
<td>$\varphi \rightarrow \varphi$</td>
</tr>
</tbody>
</table>
What if we can make assumptions?

Consider a proof for $\varphi \to \varphi$.

- Using the derivation system presented in Chapter 2, the proof takes several steps.

- But if we can make assumptions ...

\[
\begin{array}{c|c}
1 & \varphi \\
2 & \varphi \quad \text{repetition 1} \\
3 & \varphi \to \varphi \quad \text{deduction 1-2}
\end{array}
\]

This is the main idea for the deduction rule.
The deduction rule

Suppose you want to prove \( \varphi \rightarrow \psi \).
The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume $\varphi$. 

```
<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$</td>
</tr>
</tbody>
</table>
```
The **deduction** rule

Suppose you want to prove \( \varphi \rightarrow \psi \).

- Assume \( \varphi \).
- If after further steps

\[
\begin{array}{c}
\varphi \\
\vdash \\
\psi
\end{array}
\]
The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume $\varphi$.
- If after further steps
  - you can prove $\psi$, 

\[
\begin{array}{c}
\varphi \\
\hline
\vdots \\
\psi
\end{array}
\]
The **deduction** rule

Suppose you want to prove $\varphi \rightarrow \psi$.

- Assume $\varphi$.
- If after further steps you can prove $\psi$,
- then you actually have $\varphi \rightarrow \psi$.

\[
\begin{array}{c}
\varphi \\
\vdots \\
\psi \\
\hline \\
\varphi \rightarrow \psi
\end{array}
\]

deduction
Recall

The three axioms for propositional logic
Recall

The three axioms for propositional logic

1. \( \varphi \rightarrow (\psi \rightarrow \varphi) \)
Recall

The three axioms for propositional logic

1. \( \varphi \rightarrow (\psi \rightarrow \varphi) \)

2. \( (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \)
Recall

The three axioms for propositional logic

1. $\phi \rightarrow (\psi \rightarrow \phi)$

2. $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

3. $(\neg \phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \phi)$
Proving the axioms (1)

The axiom

\[ \varphi \rightarrow (\psi \rightarrow \varphi) \]

can be proved from *deduction*:
Proving the axioms (1)

The axiom

\[ \varphi \rightarrow (\psi \rightarrow \varphi) \]

can be proved from *deduction*:

1. \[ \varphi \]
The axiom

\[ \varphi \rightarrow (\psi \rightarrow \varphi) \]

can be proved from deduction:

1 \[ \varphi \]

2 \[ \psi \rightarrow \varphi \] deduction 2-3

3 \[ \varphi \] repetition 1

4 \[ \psi \] deduction 2-3

5 \[ \varphi \rightarrow (\psi \rightarrow \varphi) \] deduction 1-4
The axiom

\[ \varphi \rightarrow (\psi \rightarrow \varphi) \]

can be proved from *deduction*:

1. \( \varphi \)
2. \( \psi \)
3. \( \varphi \) repetition 1
Proving the axioms (1)

The axiom

\[ \varphi \rightarrow (\psi \rightarrow \varphi) \]

can be proved from *deduction*:

1 \[ \varphi \]
2 \[ \psi \]
3 \[ \varphi \] (repetition 1)
4 \[ \psi \rightarrow \varphi \] (deduction 2-3)
The axiom

$$\varphi \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *deduction*:

1. $$\varphi$$
2. $$\psi$$
3. $$\varphi$$\hspace{1cm} repetition 1
4. $$\psi \rightarrow \varphi$$ \hspace{1cm} deduction 2-3
5. $$\varphi \rightarrow (\psi \rightarrow \varphi)$$ \hspace{1cm} deduction 1-4
Proving the axioms (2)

The axiom

$$((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

can be proved from *modus ponens* and *deduction*:
Proving the axioms (2)

The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from *modus ponens* and *deduction*:

1. \[\varphi \rightarrow (\psi \rightarrow \chi)\]
Proving the axioms (2)

The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from *modus ponens* and *deduction*:

1. \(\varphi \rightarrow (\psi \rightarrow \chi)\)
2. \(\varphi \rightarrow \psi\)

\[\text{modus ponens 3,2}\]

\[\psi \rightarrow \chi\]

\[\text{modus ponens 3,1}\]

\[\chi\]

\[\text{modus ponens 4,5}\]

\[\varphi \rightarrow \chi\]

\[\text{deduction 3-6}\]

\[((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

\[\text{deduction 2-7}\]

\[((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)))\]

\[\text{deduction 1-8}\]
Proving the axioms (2)

The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from \textit{modus ponens} and \textit{deduction}:

1. \[\varphi \rightarrow (\psi \rightarrow \chi)\]
2. \[\varphi \rightarrow \psi\]
3. \[\varphi\]

\[\text{modus ponens 3,2}\]
\[\text{modus ponens 3,1}\]
\[\text{deduction 3-6}\]
\[\text{deduction 2-7}\]
\[\text{deduction 1-8}\]
The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from *modus ponens* and *deduction*:

1. \[\varphi \rightarrow (\psi \rightarrow \chi)\]
2. \[\varphi \rightarrow \psi\]
3. \[\varphi\]
4. \[\psi\]

\[\text{modus ponens 3,2}\]
Proving the axioms (2)

The axiom

\[(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))\]

can be proved from modus ponens and deduction:

1. \(\varphi \to (\psi \to \chi)\)
2. \(\varphi \to \psi\)
3. \(\varphi\)
4. \(\psi\)
5. \(\psi \to \chi\)

\[\text{modus ponens 3,2}\]
\[\text{modus ponens 3,1}\]
Natural Deduction for Propositional Logic

Proving the axioms (2)

The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from *modus ponens* and *deduction*:

1. \(\varphi \rightarrow (\psi \rightarrow \chi)\)
2. \(\varphi \rightarrow \psi\)
3. \(\varphi\)
4. \(\psi\)
5. \(\psi \rightarrow \chi\)
6. \(\chi\) modus ponens 3,2
7. \(\psi \rightarrow \chi\) modus ponens 3,1
8. \(\varphi \rightarrow \chi\) deduction 3-6
9. \((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)\) deduction 2-7

(\(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\)

(http://www.logicinaction.org/)
Proving the axioms (2)

The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from *modus ponens* and *deduction*:

1. \(\varphi \rightarrow (\psi \rightarrow \chi)\)
2. \(\varphi \rightarrow \psi\)
3. \(\varphi\)
4. \(\psi\)
5. \(\psi \rightarrow \chi\)
6. \(\chi\)
7. \(\varphi \rightarrow \chi\)

- modus ponens 3,2
- modus ponens 3,1
- modus ponens 4,5
- deduction 3-6
Proving the axioms (2)

The axiom

\[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\]

can be proved from *modus ponens* and *deduction*:

1. \[\varphi \rightarrow (\psi \rightarrow \chi)\]
2. \[\varphi \rightarrow \psi\]
3. \[\varphi\]
4. \[\psi\]
5. \[\psi \rightarrow \chi\]
6. \[\chi\]
7. \[\varphi \rightarrow \chi\]
8. \[(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)\]

*modus ponens 3,2*
*modus ponens 3,1*
*modus ponens 4,5*
*deduction 3-6*
*deduction 2-7*
Proving the axioms (2)

The axiom

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

can be proved from *modus ponens* and *deduction*:

1. $\varphi \rightarrow (\psi \rightarrow \chi)$
2. $\varphi \rightarrow \psi$
3. $\varphi$
4. $\psi$
5. $\psi \rightarrow \chi$  
6. $\chi$
7. $\varphi \rightarrow \chi$
8. $$(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)$$
9. $$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

- **modus ponens 3,2**
- **modus ponens 3,1**
- **modus ponens 4,5**
- **deduction 3-6**
- **deduction 2-7**
- **deduction 1-8**
We need more

The axiom

\[(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)\]

*cannot* be proved from *modus ponens* and *deduction.*
We need more

The axiom

\[(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)\]

cannot be proved from modus ponens and deduction.

We need a way to deal with negations.
The **refutation** rule

Suppose you want to prove \( \varphi \).
The **refutation** rule

Suppose you want to prove $\varphi$.

- Assume $\neg \varphi$.

$$\underline{\neg \varphi}$$
The **refutation** rule

Suppose you want to prove $\varphi$.

- Assume $\neg \varphi$.
- If after further steps

```
<table>
<thead>
<tr>
<th></th>
<th>\neg \varphi</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

...
The **refutation** rule

Suppose you want to prove $\varphi$.

- Assume $\neg \varphi$.
- If after further steps you can prove a contradiction $\bot$, 

\[
\begin{array}{c}
\neg \varphi \\
\vdash \bot
\end{array}
\]
The **refutation** rule

Suppose you want to prove $\varphi$.

- Assume $\neg \varphi$.
- If after further steps you can prove a contradiction $\bot$, then $\neg \varphi$ cannot be true

```
  \neg \varphi
  
  \vdash \bot
```
The **refutation** rule

Suppose you want to prove \( \varphi \).
- Assume \( \neg \varphi \).
- If after further steps you can prove a contradiction \( \bot \),
- then \( \neg \varphi \) cannot be true
- so you actually have \( \varphi \).

\[
\begin{array}{c}
\neg \varphi \\
\vdots \\
\bot \\
\hline
\varphi
\end{array}
\]
Proving the axioms (3)

The axiom

$$(\neg \varphi \to \neg \psi) \to (\psi \to \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1. $\neg \varphi \to \neg \psi$
2. $\psi$
3. $\neg \varphi$
4. $\neg \psi$
5. $\bot$ (modus ponens 3,1)
6. $\varphi$ (modus ponens 2,4)
7. $\psi \to \varphi$ (refutation 3-5)
8. $$(\neg \varphi \to \neg \psi) \to (\psi \to \varphi)$$ (deduction 1-7)
The axiom

\[(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)\]

can be proved from *modus ponens*, *deduction* and *refutation*:

1. \(\neg \varphi \rightarrow \neg \psi\)

For step 5, note that \(\neg \psi\) can be seen as an abbreviation of \(\psi \rightarrow \bot\).
Proving the axioms (3)

The axiom

\[(\neg \varphi \to \neg \psi) \to (\psi \to \varphi)\]

can be proved from *modus ponens*, *deduction* and *refutation*:

1. \(\neg \varphi \to \neg \psi\)
2. \(\psi\)

For step 5, note that \(\neg \psi\) can be seen as an abbreviation of \(\psi \to \bot\).
Proving the axioms (3)

The axiom

\[(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)\]

can be proved from *modus ponens, deduction* and *refutation*:

1. \(\neg \varphi \rightarrow \neg \psi\)
2. \(\psi\)
3. \(\neg \varphi\)

For step 5, note that \(\neg \psi\) can be seen as an abbreviation of \(\psi \rightarrow \bot\).
Proving the axioms (3)

The axiom

\[(\neg \varphi \to \neg \psi) \to (\psi \to \varphi)\]

can be proved from *modus ponens*, *deduction* and *refutation*:

1. \(\neg \varphi \to \neg \psi\)
2. \(\psi\)
3. \(\neg \varphi\)
4. \(\neg \psi\)
5. \(\bot\) \text{ modus ponens 3,1}
6. \(\psi\) \text{ modus ponens 2,4}
7. \(\varphi\) \text{ refutation 3-5}
8. \((\neg \varphi \to \neg \psi) \to (\psi \to \varphi)\) \text{ deduction 1-7}

For step 5, note that \(\neg \psi\) can be seen as an abbreviation of \(\psi \to \bot\).
Proving the axioms (3)

The axiom

\[(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)\]

can be proved from *modus ponens*, *deduction* and *refutation*:

\[
\begin{array}{c|c}
1 & \neg \varphi \rightarrow \neg \psi \\
2 & \psi \\
3 & \neg \varphi \\
4 & \neg \psi \\
5 & \bot \\
\end{array}
\]

- modus ponens 3,1
- modus ponens 2,4

For step 5, note that \(\neg \psi\) can be seen as an abbreviation of \(\psi \rightarrow \bot\).
Proving the axioms (3)

The axiom

\[(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)\]

can be proved from *modus ponens*, *deduction* and *refutation*:

1. \[\neg \varphi \rightarrow \neg \psi\]
2. \[\psi\]
3. \[\neg \varphi\]
4. \[\neg \psi\]  
   *modus ponens* 3,1
5. \[\bot\]  
   *modus ponens* 2,4
6. \[\varphi\]  
   *refutation* 3-5

For step 5, note that \[\neg \psi\] can be seen as an abbreviation of \[\psi \rightarrow \bot\].
Proving the axioms (3)

The axiom

\[(\neg \varphi \to \neg \psi) \to (\psi \to \varphi)\]

can be proved from modus ponens, deduction and refutation:

\begin{align*}
1 &: \neg \varphi \to \neg \psi \\
2 &: \psi \\
3 &: \neg \varphi \\
4 &: \neg \psi \\
5 &: \bot \\
6 &: \varphi \\
7 &: \psi \to \varphi
\end{align*}

modus ponens 3,1

modus ponens 2,4

refutation 3-5

deduction 2-6
Natural Deduction for Propositional Logic

Proving the axioms (3)

The axiom

$$(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1. $\neg \varphi \rightarrow \neg \psi$
2. $\psi$
3. $\neg \varphi$
4. $\neg \psi$
5. $\bot$ modus ponens 3,1
6. $\varphi$ modus ponens 2,4
7. $\psi \rightarrow \varphi$ refutation 3-5
8. $$(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)$$ deduction 1-7
The axiom

$$(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)$$

can be proved from *modus ponens*, *deduction* and *refutation*:

1. $\neg \varphi \rightarrow \neg \psi$
2. $\psi$
3. $\neg \varphi$
4. $\neg \psi$
5. $\bot$
6. $\varphi$
7. $\psi \rightarrow \varphi$
8. $$(\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi)$$

For step 5, note that $\neg \psi$ can be seen as an abbreviation of $\psi \rightarrow \bot$. 

(http://www.logicinaction.org/)
So ...

\[ \varphi, \varphi \rightarrow \psi \]

\[ \varphi \quad \varphi \rightarrow \psi \quad \psi \quad \text{modus ponens} \]

\[ \varphi \quad \vdash \psi \quad \text{deduction} \]

\[ \neg \varphi \quad \vdash \bot \quad \text{refutation} \]

(www.logicinaction.org/)
So ...

The *modus ponens*, *deduction* and *refutation* rules are a complete system for propositional logic.

\[
\begin{align*}
\phi, \phi \rightarrow \psi \\
\hline
\psi \\
\end{align*}
\]

(modus ponens)

\[
\begin{array}{c}
\phi \\
\hline
\psi \\
\end{array}
\]

(deduction)

\[
\begin{array}{c}
\neg \phi \\
\hline
\bot \\
\end{array}
\]

(refutation)
To facilitate things ...
To facilitate things . . .

- Natural deduction introduces rules to manipulate all the connectives in an easy way.
For implication $\rightarrow$
For implication $\rightarrow$

\[ \varphi, \varphi \rightarrow \psi \quad \text{modus ponens} \]

\[ \psi \]
For implication →

\[ \varphi, \varphi \rightarrow \psi \]

\[ \psi \]

\[ \text{modus ponens} \]

\[ \text{E}_\rightarrow \]
For implication $\rightarrow$

\[ \varphi, \varphi \rightarrow \psi \]

\[ \psi \]

modus ponens

\[ \varphi \]

\[ \vdots \]

\[ \psi \]

\[ \varphi \rightarrow \psi \]

deduction

$E_{\rightarrow}$
For implication $\rightarrow$

$\varphi, \varphi \rightarrow \psi$

$\varphi \rightarrow \psi$

$\psi$

modus ponens

$\varphi$

$\vdots$

$\psi$

$\varphi \rightarrow \psi$

deduction

E$

I$
For negation ¬
For negation $\neg$

\[
\neg \varphi, \varphi
\]

$\bot$

For negation \( \neg \)

\[
\begin{align*}
\neg \varphi, \varphi \\
\hline
\bot \\
\end{align*}
\]

\[\text{E}_\neg\]
For negation $\neg$
For negation \( \neg \)

\[
\begin{array}{c}
\neg \varphi, \varphi \\
\hline
\bot
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\hline
\bot
\end{array}
\]

\[
\begin{array}{c}
\neg \varphi
\end{array}
\]

\[
\begin{array}{c}
\bot
\end{array}
\]

refutation

\[
\begin{array}{c}
\varphi
\end{array}
\]

\[
\begin{array}{c}
E_{\neg}
\end{array}
\]

\[
\begin{array}{c}
I_{\neg}
\end{array}
\]

(http://www.logicinaction.org/)
For conjunction ∧
For conjunction $\land$

$\frac{\varphi \land \psi}{\varphi}$

$\frac{\varphi \land \psi}{\psi}$
For conjunction $\land$

\[
\begin{align*}
\phi \land \psi \\
\hline
\phi \\
\hline
\phi \land \psi \\
\hline
\psi \\
\hline
E_{\land}
\end{align*}
\]
For conjunction $\land$

\[
\begin{array}{c}
\varphi \land \psi \\
\varphi \\
\varphi \land \psi \\
\psi \\
\hline \\
E_{\land} \\
\end{array}
\]

\[
\begin{array}{c}
\varphi, \psi \\
\varphi \land \psi \\
\end{array}
\]
For conjunction $\land$

\[
\begin{array}{c}
\frac{\varphi \land \psi}{\varphi} \\
\frac{\varphi \land \psi}{\psi}
\end{array}
\quad
\begin{array}{c}
\frac{\varphi, \psi}{\varphi \land \psi}
\end{array}
\]

$E_{\land}$  $I_{\land}$
For disjunction $\lor$
For disjunction $\lor$

\[ \varphi \lor \psi, \quad \vdash \varphi, \quad \vdash \psi \]

\[ \vdash \chi \]
For disjunction $\lor$

\[\varphi \lor \psi, \quad \varphi, \quad \psi \rightarrow \chi, \quad \chi, \quad \text{E}_\lor\]
For disjunction $\lor$

\[
\begin{array}{c}
\phi \lor \psi, \\
\vdots, \\
\chi \\
\hline
\phi \\
\psi \\
\hline
\chi \\
\hline
\phi \lor \psi
\end{array}
\]

$E_\lor$

\[
\begin{array}{c}
\phi \\
\hline
\phi \lor \psi
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\hline
\phi \lor \psi
\end{array}
\]

For disjunction $\lor$

\[
\phi \lor \psi, \\
\vdots, \\
\chi \\
\hline
\phi \\
\psi \\
\hline
\chi \\
\hline
\phi \lor \psi
\]

$E_\lor$

\[
\begin{array}{c}
\phi \\
\hline
\phi \lor \psi
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\hline
\phi \lor \psi
\end{array}
\]

For disjunction $\lor$

\[
\phi \lor \psi, \\
\vdots, \\
\chi \\
\hline
\phi \\
\psi \\
\hline
\chi \\
\hline
\phi \lor \psi
\]

$E_\lor$

\[
\begin{array}{c}
\phi \\
\hline
\phi \lor \psi
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\hline
\phi \lor \psi
\end{array}
\]

For disjunction $\lor$

\[
\phi \lor \psi, \\
\vdots, \\
\chi \\
\hline
\phi \\
\psi \\
\hline
\chi \\
\hline
\phi \lor \psi
\]

$E_\lor$

\[
\begin{array}{c}
\phi \\
\hline
\phi \lor \psi
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\hline
\phi \lor \psi
\end{array}
\]

For disjunction $\lor$

\[
\phi \lor \psi, \\
\vdots, \\
\chi \\
\hline
\phi \\
\psi \\
\hline
\chi \\
\hline
\phi \lor \psi
\]

$E_\lor$

\[
\begin{array}{c}
\phi \\
\hline
\phi \lor \psi
\end{array}
\]

\[
\begin{array}{c}
\psi \\
\hline
\phi \lor \psi
\end{array}
\]
For disjunction $\lor$

\[
\varphi \lor \psi, \\
\vdots, \\
\chi
\]

\[
\psi \\
\vdots \\
\chi
\]

$\chi$

$\varphi \lor \psi$

$\psi$

$\varphi \lor \psi$

$\varphi \lor \psi$

$\chi$

$\chi$

$\chi$

$I_\lor$

$E_\lor$
In order to present *introduction* and *elimination* rules for both ∀ and ∃, we need to recall two notions.
For *predicate* logic

In order to present *introduction* and *elimination* rules for both $\forall$ and $\exists$, we need to recall two notions.

- **Bounded variable.**
For predicate logic

In order to present *introduction* and *elimination* rules for both $\forall$ and $\exists$, we need to recall two notions.

- **Bounded variable.**

- **Substitution of a variable for a term in a formula.**
Bounded variable

Scope of a quantifier. In a formula of the form \( \forall x \varphi \) (\( \exists x \varphi \)), the subformula \( \varphi \) is said to be the scope of the quantifier \( \forall \) (\( \exists \)).

Binding a variable. In a formula of the form \( \forall x \varphi \) (\( \exists x \varphi \)), the quantifier \( \forall \) (\( \exists \)) binds any occurrence of \( x \) in \( \varphi \) that is not bounded by another quantifier inside \( \varphi \).

Bounded variable. An occurrence of a variable \( x \) is bounded in a formula \( \varphi \) if there is a quantifier in \( \varphi \) that binds it.
Bounded variable

- **Scope of a quantifier.** In a formula of the form $\forall x \varphi \ (\exists x \varphi)$, the subformula $\varphi$ is said to be the **scope** of the quantifier $\forall (\exists)$. 
Bounded variable

- **Scope of a quantifier.** In a formula of the form $\forall x \varphi \ (\exists x \varphi)$, the subformula $\varphi$ is said to be the **scope** of the quantifier $\forall \ (\exists)$.

- **Binding a variable.** In a formula of the form $\forall x \varphi \ (\exists x \varphi)$, the quantifier $\forall \ (\exists)$ **binds** any occurrence of $x$ in $\varphi$ that is not bounded by another quantifier inside $\varphi$.
Bounded variable

- **Scope of a quantifier.** In a formula of the form $\forall x \varphi (\exists x \varphi)$, the subformula $\varphi$ is said to be the **scope** of the quantifier $\forall (\exists)$.

- **Binding a variable.** In a formula of the form $\forall x \varphi (\exists x \varphi)$, the quantifier $\forall (\exists)$ **binds** any occurrence of $x$ in $\varphi$ that is not bounded by another quantifier inside $\varphi$.

- **Bounded variable.** An occurrence of a variable $x$ is **bounded** in a formula $\varphi$ if there is a quantifier in $\varphi$ that binds it.
Substitution (1)

Substitution inside a term. Replacing the occurrences of the variable $y$ for the term $t$ inside the term $s$ produces the term denoted by $(s)^y_t$.

Formally,

For a constant:

$$(c)^y_t := c$$

For a variable:

$$(x)^y_t := x \text{ for } x \text{ different from } y$$

$$(y)^y_t := t$$

Examples:

$$(a)^x_c := a$$

$$(x)^z_a := x$$

$$(z)^y := y$$

(http://www.logicinaction.org/)
Substitution (1)

- **Substitution inside a term.** Replacing the occurrences of the variable \( y \) for the term \( t \) inside the term \( s \) produces the term denoted by

\[(s)^y_t\]
Substitution (1)

- **Substitution inside a term.** Replacing the occurrences of the variable $y$ for the term $t$ inside the term $s$ produces the term denoted by

$$ (s)_t^y $$

- Formally,

For a constant:

$$ (c)_t^y := c $$

For a variable:

$$ \begin{cases} 
(x)_t^y := x & \text{for } x \text{ different from } y \\
(y)_t^y := t 
\end{cases} $$

Examples:

$$ (a)_c^x := a $$

$$ (x)_a^z := z $$

$$ (y)_y^t := t $$

(http://www.logicinaction.org/)
Substitution (1)

- **Substitution inside a term.** Replacing the occurrences of the variable $y$ for the term $t$ inside the term $s$ produces the term denoted by $$(s)^y_t$$

- Formally,
  
  For a **constant**:
  
  $$(c)^y_t := c$$

  For a **variable**:
  
  $$\begin{cases} 
  (x)^y_t := x & \text{for } x \text{ different from } y \\
  (y)^y_t := t & 
  \end{cases}$$

Examples:

- $$(a)^x_c := a$$
- $$(x)^y_a := x$$
- $$(z)^z_y := y$$
Substitution (2)

- Substitution inside a formula
- Replacing the free occurrences of the variable $y$ for the term $t$ inside the formula $\phi$ produces the formula denoted by $(\phi)_y^t$

Formally,

\[
(P_{t_1} \cdots t_n)_y^t := P_{(t_1)_y^t} \cdots (t_n)_y^t.
\]

\[
(\neg \phi)_y^t := \neg (\phi)_y^t.
\]

\[
(\phi \land \psi)_y^t := (\phi)_y^t \land (\psi)_y^t.
\]

\[
(\phi \lor \psi)_y^t := (\phi)_y^t \lor (\psi)_y^t.
\]

\[
(\phi \rightarrow \psi)_y^t := (\phi)_y^t \rightarrow (\psi)_y^t.
\]

\[
(\phi \leftrightarrow \psi)_y^t := (\phi)_y^t \leftrightarrow (\psi)_y^t.
\]

\[
(\forall x \phi)_y^t := \forall x (\phi)_y^t.
\]

\[
(\exists y \phi)_y^t := \exists y (\phi)_y^t.
\]
Substitution (2)

Substitution inside a formula. Replacing the free occurrences of the variable \( y \) for the term \( t \) inside the formula \( \varphi \) produces the formula denoted by

\[(\varphi)^y_t\]
Substitution inside a formula. Replacing the free occurrences of the variable $y$ for the term $t$ inside the formula $\varphi$ produces the formula denoted by

$$(\varphi)^y_t$$

Formally,

$$(Pt_1 \cdots t_n)^y_t := P(t_1)^y_t \cdots (t_n)^y_t$$

$$(-\varphi)^y_t := - (\varphi)^y_t$$

$$(\varphi \land \psi)^y_t := (\varphi)^y_t \land (\psi)^y_t$$

$$(\varphi \lor \psi)^y_t := (\varphi)^y_t \lor (\psi)^y_t$$

$$(\varphi \rightarrow \psi)^y_t := (\varphi)^y_t \rightarrow (\psi)^y_t$$

$$(\varphi \leftrightarrow \psi)^y_t := (\varphi)^y_t \leftrightarrow (\psi)^y_t$$

$$\{$$

$$((\forall x \varphi)^y_t := \forall x (\varphi)^y_t$$

$$((\forall y \varphi)^y_t := \forall y \varphi$$

$$((\exists x \varphi)^y_t := \exists x (\varphi)^y_t$$

$$((\exists y \varphi)^y_t := \exists y \varphi$$

$$\}$$
For the universal quantifier $\forall$
For the universal quantifier $\forall$

\[ \forall x \varphi \]

\[ (\varphi)_t^x \]

provided that no variable in $t$
occurs bounded in $\varphi$
For the universal quantifier $\forall$

\[ \forall x \varphi \]

\[ \frac{\varphi}{(\varphi)_t} \]

provided that no variable in $t$
occurs bounded in $\varphi$

\[ E_{\forall} \]
For the universal quantifier $\forall$

\[
\begin{align*}
\forall x \varphi \\
\hline
(\varphi)_t \\
\hline
(\varphi)_u \\
\hline
\forall x \varphi
\end{align*}
\]

provided that no variable in $t$ occurs bounded in $\varphi$

for $u$ a special symbol not used anywhere else in the proof

$E_{\forall}$
For the universal quantifier $\forall$

$\forall x \varphi$  

$(\varphi)_t^x$  

provided that no variable in $t$ occurs bounded in $\varphi$

$\forall x \varphi$  

$(\varphi)_u^x$  

for $u$ a special symbol not used anywhere else in the proof

$E_\forall$  

$I_\forall$
For the existential quantifier $\exists$
For the existential quantifier \( \exists \)

\[
\begin{align*}
\exists x \varphi, & \quad (\varphi)_u^x \\
\vdash & \\
\psi & \\
\psi & \\
\end{align*}
\]

for \( u \) a special symbol not used anywhere in the proof
For the existential quantifier $\exists$

$\exists x \varphi$,  

\[
\begin{array}{c}
  \exists x \varphi, \\
  \hline
  (\varphi)^u_x \\
  \vdots \\
  \psi
\end{array}
\]

$\psi$

for $u$ a special symbol not used anywhere in the proof

$E_\exists$
For the existential quantifier $\exists$

$\exists x \varphi , \vdash \psi$

provided that no variable in $t$ occurs bounded in $\varphi$

$\text{E}_\exists$

for $u$ a special symbol not used anywhere in the proof
For the existential quantifier $\exists$

For the existential quantifier $\exists$

$$\exists x \varphi, \quad (\varphi)_u^x \quad \vdash \psi$$

$$\vdash \psi$$

for $u$ a special symbol not used anywhere in the proof

provided that no variable in $t$ occurs bounded in $\varphi$

$E_\exists$ $I_\exists$
For the identity symbol $=$
For the identity symbol $\equiv$

\[
\begin{align*}
& t_1 = t_2, \varphi \\
\quad \varphi[t_1/t_2] \\
& t_1 = t_2, \varphi \\
\quad \varphi[t_2/t_1]
\end{align*}
\]

where $\varphi[t_1/t_2]$ is the result of replacing, in $\varphi$, some occurrences of $t_2$ by $t_1$, provided that...
For the identity symbol $=\!$ 

\[
\begin{array}{c}
  t_1 = t_2, \varphi \\
  \varphi[t_1/t_2] \\
  t_1 = t_2, \varphi \\
  \varphi[t_2/t_1]
\end{array}
\]

where $\varphi[t_1/t_2]$ is the result of replacing, in $\varphi$, some occurrences of $t_2$ by $t_1$, provided that

- $t_2$ contains only variables that occur freely in $\varphi$, and
For the identity symbol $\equiv$:

$$
\begin{align*}
  t_1 &= t_2, \varphi \\
  \varphi[t_1/t_2] \\
  t_1 &= t_2, \varphi \\
  \varphi[t_2/t_1]
\end{align*}
$$

where $\varphi[t_1/t_2]$ is the result of replacing, in $\varphi$, some occurrences of $t_2$ by $t_1$, provided that:

- $t_2$ contains only variables that occur freely in $\varphi$, and
- $t_1$ contains only variables that do not get bounded after replacement.
For the identity symbol $=$

\[
\begin{align*}
\frac{t_1 = t_2 \; , \; \varphi}{\varphi[t_1/t_2]} \\
\frac{t_1 = t_2 \; , \; \varphi}{\varphi[t_2/t_1]}
\end{align*}
\]

where $\varphi[t_1/t_2]$ is the result of replacing, in $\varphi$, some occurrences of $t_2$ by $t_1$, provided that

- $t_2$ contains only variables that occur freely in $\varphi$, and
- $t_1$ contains only variables that do not get bounded after replacement.

$E_=$
For the identity symbol $=\ $ \\

\[
\frac{t_1 = t_2, \varphi}{\varphi[t_1/t_2]}
\]
\[
\frac{t_1 = t_2, \varphi}{\varphi[t_2/t_1]}
\]

where $\varphi[t_1/t_2]$ is the result of replacing, in $\varphi$, some occurrences of $t_2$ by $t_1$, provided that

- $t_2$ contains only variables that occur freely in $\varphi$, and
- $t_1$ contains only variables that do not get bounded after replacement.

for any term $t$. 

\[\text{E=}\]
For the identity symbol $\equiv$

\[
\begin{align*}
t_1 &= t_2, \varphi \\
\varphi[t_1/t_2] &\quad \text{for any term } t.
\end{align*}
\]

where $\varphi[t_1/t_2]$ is the result of replacing, in $\varphi$, some occurrences of $t_2$ by $t_1$, provided that

- $t_2$ contains only variables that occur freely in $\varphi$, and
- $t_1$ contains only variables that do not get bounded after replacement.

\[
E_\equiv \\
I_\equiv
\]